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Heated Region Around Boreholes

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ABSTRACT

The heat transfer mechanism in the rock mass around a borehole is an axisymmetric heat conduction process. Earlier works considered this phenomenon to be a two-dimensional radial heat flow, which is congruent in any horizontal plane. The solution is obtained by solving the differential equation of the heat conduction, applying a cylindrical coordinate system. Accordingly a cylindrical interface is the boundary between the heated and the undisturbed rock mass.

Now a new method will be applied in which the heat flux field is the function determined primarily, instead of the temperature field. The axisymmetric heat flux field can be described by complex-variable analytic functions. Because of the validity of the Cauchy-Riemann equations the particular solutions of them can be superimposed. In our model the terrestrial heat flux is a homogeneous component, to which a line-source of variable intensity is placed as a singularity, distorting the homogeneous heat flux field.

The heat flux field can be determined analogously to an axisymmetric perfect fluid flow. The isotherms of the temperature filed forms a set of axisymmetric surfaces, analogously to the velocity potential, orthogonally to the stream surfaces which are tangential to the heat flux vectors. The thermal potential function and the heat-stream function are determined by potential-theory method. The stream surface determined by the zero constant, divides the heated region from the intact rock mass. This domain is a paraboloid-like body of revolution around the well axis. Its equation is obtained by analytic method. The equation of the bounding surface is a complex transcendent expression, can be determined numerically only.

This method is more accurate to describe and calculate the borehole heat transfer.

1.INTRODUCTION

The wellbore heat transfer is among the phenomena most important to engineers in geothermics. It has been the object of much attention in the last half century. There have been several theoretical and experimental studies to determine the temperature distribution of the flowing fluid along the depth in geothermal wells.

BOLDIZSÁR (1958) was the first to make a thorough attack on the borehole heat transfer problem. His theory was based on the transient heat conduction equation written for the rock mass around the well. The obtained parabolic differential equation was transformed, applying a Laplace-transformation, into a Bessel-type differential equation. The solution was obtained in terms of Bessel functions. The thermal resistance between the fluid and the adjacent rock was neglected.

RAMEY (1962) was made same simplifications on BOLDIZSÁR's solution. However the overall heat transfer coefficient was taken into consideration. He thus derived an ordinary, first order, inhomogeneous differential equation. Its solution has got some approximations: i.e. the well completion is homogeneous along the depth, the overall heat transfer coefficient and the transient heat conduction functions are considered to be uniform.

Several authors: WILLHITE (1968), BOBOK (1987), TÓTH (2002) have completed RAMEY's method by more sophisticated details. This paper is focused to a substantial element of RAMEY's solution: the transient heat conduction function. It is described here a computational method which shows that this function depends on the depth. In the succeeding sections a new mathematical approach is introduced, which is motivated by the two-dimensional method of thermal singularities BOBOK (1981). The present study is the generalized version, for three-dimensional, axisymmetric case.

2. BASIC METHOD

As it is well-known, the temperature distribution of the up-flowing fluid in the tubing can be obtained by RAMEY's method as:

$$T = T_0 + \gamma(z + A) - \gamma A e^{\frac{z-H}{A}} \tag{1}$$

in which T_0 is the surface temperature

γ is the geothermal gradient

z is the depth coordinate

H is the bottom-hole depth

$$A = \frac{\dot{m} c(k + f \cdot R_{Ti} \cdot U_{Ti})}{2\pi R_{Ti} U_{Ti} k} \tag{2}$$

where $\dot{m}$ is the mass flow rate of the fluid

c is the specific heat capacity of it

k is the heat conductivity of the rock

R_{Ti} is the internal radius of the tubing

U_{Ti} is the overall heat transfer coefficient referring to R_{Ti}

f is the transient heat conduction function.

In steady state heat conduction the overall heat flux is

$$\dot{Q} = 2\pi k \frac{T_w - T_0 - \gamma z}{\ln \frac{R_\infty}{R_w}} \quad (3)$$

Where T_w is the temperature at the borehole wall

R_w is the radius of the borehole

R_∞ is the radius of the heated region around the well.

Following this formulation of the problem, the assumption, that the transient heat transfer function doesn't depend on the depth is equivalent with the statement that the radius of the heated region are the same at any depth i.e. the bounding surface of the heated region is a cylindrical surface. Instead of this, in the following it will be demonstrated that this boundary surface is a surface of rotation around a vertical axis.

3. METHOD OF THERMAL SINGULARITIES

The main feature of the method of singularities is that, instead of the solution of the heat conduction equation, the heat flux vector field is primarily determined. The heat flux vector is obtained by FOURIER's law

$$\vec{q} = -k \, \mathrm{grad} T \quad (4)$$

Assuming uniform heat conductivity we get

$$\vec{q} = \mathrm{grad}(-kT) = \mathrm{grad} U \quad (5)$$

In this case U=-kT can be considered as a thermal potential function. Its existence yield

$$\mathrm{rot}\, \vec{q} \equiv 0 \quad (6)$$

If the heat conduction is steady and assuming no heat sources, the heat flux field is solenoidal:

$$\mathrm{div}\, \vec{q} \equiv 0 \quad (7)$$

These latter two equations for two-dimensional flow are

$$\frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y} \equiv 0 \quad (8)$$

and

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \equiv 0 \quad (9)$$

From Eq. (5), it follows

$$q_x = \frac{\partial U}{\partial x} \qquad q_y = \frac{\partial U}{\partial y} \quad (10)$$

Eq.(9) is fulfilled, if the expressions

$$q_x = \frac{\partial V}{\partial y} \qquad q_y = -\frac{\partial V}{\partial x} \quad (11)$$

are valid. The harmonic function V is the so-called heat flux stream function. This can be proven easily: along the curves V=const, it is obvious

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy = 0 \quad (12)$$

Substituting Eqs.(13) into (14) we get

$$-q_y dx + q_x dy = 0 \quad (13)$$

From which it follows that

$$\frac{dx}{q_x} = \frac{dy}{q_y} \quad (14)$$

and we can recognized that V=const. curves are everywhere tangent to the heat flux vectors. From Eqs.(12) and (13) it fellows that

$$q_x = \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad q_y = \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \quad (15)$$

These equations are called CAUCHY-RIEMANN equations. The necessary and sufficient conditions for complex variable function W to be analytic are that they satisfy the CAUCHY-RIEMANN equations let it be differentiable and single-valued.

Thus each two-dimensional heat flow pattern corresponds, a complex variable function W(z) in which the real part is the thermal potential U(x,y) and the imaginary part is the heat flux stream function V(x,y) as

$$W(z) = U(x, y) + iV(x, y) \quad (16)$$

In heat conduction problems we must generally restrict W to the class of analytic function: both the function W(z) and its derivate $\frac{dW}{dz}$ are single valued and finite.

In this case W(z) is called to the complex potential of the heat flux filed. Of course there can be points on the z plane, where the W(z) function is not analytic, but it is analytic at every point in the plane, than such a point is called singular point as a singularity of the function. For example, if

$$W = K \ln z \quad (17)$$

where K is a real constant, the function is analytic at every point except of the point z=0, where it is discontinuous, hence z=0 is singular point. The mathematical singularity corresponds always a thermal singularity. In the above example at the point z=0 there is a heat source. It can be show easier that $Q = K2\pi$, where Q is called the strength of source. If Q<0 it is a sink. Any singularities induce a heat flow pattern around it. Although singularities are simply mathematical convenience, they are great practical value in that, when combined with other simple heat flow patterns they can reproduce closely many complicated natural heat flux patterns. The possibility of the superposition stems from the linearity of LAPLACE's equation. If we add together a number of complex potentials the sum of these will satisfy LAPLACE's equation.

The terrestrial heat flux has the same regional interior. Its complex potential can be written as

$$W = q_\infty z \tag{18}$$

obviously satisfies the LAPLACE's equation. Combination of this potential with the potential of singularities can produce practically important heat flux patterns. The real part of the potential lines determine the isotherms in the plane. The imaginary part of the complex potential obtains the heat flux streamlines

$$\mathrm{Re}(W) = U(x,y) = \mathrm{const} \tag{19}$$

$$\mathrm{Im}(W) = V(x,y) = \mathrm{const} \tag{20}$$

Finally the derivative the complex potential W is the conjugate the heat flux vector.

$$\vec{q} = q_x - iq_y = \frac{dW}{dz} \tag{21}$$

Applying the method of singularities for two-dimensional heat conduction problems we can obtain solutions relative easily even for complicated boundary conditions.

The greatest advance of the method of singularities is that it can be expanded for three-dimensional axisymmetric case. The thermal potential function U satisfies the LAPLACE's equation. It can be written in cylindrical coordinates as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{22}$$

For axisymmetric heat flux vector field

$$q_r = q_r(z,r) \; q_\phi = 0 \qquad q_z = q_z(r,z) \tag{23}$$

In this case

$$q_r = \frac{dU}{dr} \qquad q_z = \frac{\partial U}{\partial z} \tag{24}$$

The LAPLACE equation for this case is

$$\frac{\partial^2 U}{\partial r^2} + \frac{\partial^2 U}{\partial z^2} + \frac{1}{r}\frac{dU}{dr} = 0 \tag{25}$$

Analogously to the two-dimensional case a heat flux stream function can be interpreted. The stream surfaces are adiabatic boundary surfaces, tangented by the heat flux vectors.

In this case the component of the heat flux are

$$q_r = \frac{1}{r}\frac{\partial V}{\partial z} \qquad q_z = -\frac{1}{r}\frac{\partial V}{\partial r} \tag{26}$$

Substituting these to Eq. (6) the stream function satisfies the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{r}\frac{\partial V}{\partial r} = 0 \tag{27}$$

This isn't the LAPLACE's equation yet, but its structure is quite similar, the sign of the third term negative only. Thus in three-dimensional case there is no complex potential of the heat flow. But the thermal potential exists, its constant values determined potential surfaces i.e. isotherms. The heat flux components are obtained by a simple differentiation. Complicated heat flow patterns can be constructed by superimposing elementary thermal potential functions.

The heat flux field around a production well is influenced by two effects. The first is the terrestrial heat flux, flowing vertically upward with a uniform strength. Its potential can be written as

$$U = q_\infty z \tag{28}$$

The other is the well, heating the surrounding rock mass with a radially outward flux. The strength of this flux depends on the temperature difference between the upflowing water and the surrounding rock. This difference is zero at the bottom of the well and has its maximum at the wellhead. We may assume approximately, that this temperature difference linearly increases from the bottomhole to the wellhead. The material thermal inhomogeneity can be replaced in the mathematical model by a singularity: a line source of varying strength.

The potential of this line source can be written as

$$U = -\frac{1}{4\pi}\int_{\xi=0}^{H} \sigma(\xi)\frac{d\xi}{\sqrt{r^2 + (z-\xi)^2}} \tag{29}$$

in which $\sigma(\xi)$ is the strength of the line source,

ξ is the running point of the integration,

H is the bottomhole depth of the well.

Thus the axisymmetric potential heat flow around the well can be approximated superimposing the potential of a line source of varying strength on that of the uniform terrestrial heat flow. The combined thermal potential can be expressed in the form.

$$U(r,z) = q_\infty z - \int_{\xi=0}^{H} \sigma(\xi)\frac{d\xi}{\sqrt{r^2 + (z-\xi)^2}} \tag{30}$$

The stream function is obtained from the Eqs. (7) and (26):

$$V(r,z) = q_\infty \frac{r^2}{2} + \frac{1}{4\pi}\int_{\xi=0}^{H}\left[\sigma(\xi)\frac{1}{\sqrt{r^2 + (z-\xi)^2}} - 1\right] d\xi \tag{31}$$

The strength of the line source can be determined applying an approximate assumption.

The overall heat flux in the rock around the well is actuated by the temperature difference between the borehole wall and the undisturbed rock mass. The temperature of the borehole wall can be hardly determined. Let's consider Fig.1. The temperature of the flowing fluid differs, only a few centigrades from the bottomhole temperature.

$$T_b = T_0 + \gamma H \quad (32)$$

The temperature at the outer radius of the cement sheet of the well is closer to T_b, than the fluid temperature. Thus, instead of the temperature difference $T_w - T_0 - \gamma z$, the difference $T_b - T_0 - \gamma z$ is taken. The overall heat flux for a layer of unit thickness is obtained as

$$Q = 2\pi k \frac{\gamma(H - z)}{\ln \frac{R_\infty}{R_w}} \quad (33)$$

Since

$$\xi = H - z \quad (34)$$

it can be written, that

$$\sigma = C\xi \quad (35)$$

Thus the strength of the line source is a simple linear function which can be integrated easily. It is evident that the surface V(r,z) equal to a sequence of constants are all point tangent to the heat flux vectors and hence define adiabatic surfaces of perfect heat isolation. The stream surface determined by the equation V=0 is that adiabatic surface which forms the contour of the heated region around the well. Thus we got the expression

$$q_\infty \frac{r^2}{2} + \frac{1}{4\pi} \int_{\xi=0}^{H} \left(\frac{1}{\sqrt{r^2 + (z - \xi)^2}} - 1 \right) \cdot C \cdot \xi \cdot d\xi = 0$$
(36)

After integration a transcendent equation is obtained

$$\frac{q_\infty r^2}{2} - \frac{C}{8\pi} \ln \frac{r^2 + (z-H)^2}{r^2 + z^2} + \frac{C \cdot r}{4\pi} \left(\operatorname{arctg} \frac{z-H}{r} - \operatorname{arctg} \frac{z}{r} \right) - \frac{1}{8\pi} C \cdot H^2 = 0$$
(37)

The solution of this equation can be possible numerically only. A sequence of z values is taken and in each depth the equation is solved for r. It was carried out applying the method of MATLAB FZERO. To the obtained points a polynome was fitted by the method of MATLAB PLOYFIT. The curve had been determined by this procedure was rotated around the coordinate axis z. This leads to the equation of the surface of rotation

$$z(r) = z\left(\sqrt{x^2 + y^2}\right) \quad (38)$$

The obtained surface is shown in Fig.2. It is substantially different than the earlier assumed circular cylindrical contour.

4. EXPERIMENTAL RESULTS

The contour of the heated region can be determined by experiments too. It is obtained by RAMEY's solution that

$$A \frac{dT}{dz} = T - T_0 - \gamma z \quad (39)$$

The coefficient A can be determined knowing the measured temperature distribution of given well. It is shown in Fig.3. The coefficient A_i can be calculated by finite differences:

$$A_i = \frac{T_i - T_0 - \gamma z_i}{\frac{T_{i+1} - T_i}{z_{i+1} - z_i}} \quad (40)$$

Expressing the transient heat conduction function from Eq.(2) we get

$$f = 2\pi A k - \dot{m} c \frac{k}{R_{1B} U_{1B}} \quad (41)$$

For steady heat conduction

$$f = \ln \frac{R_\infty}{R_w} \quad (42)$$

Thus the radius of the contour of the heated region is obtained as

$$R_\infty = R_w e^{f(z)} \quad (43)$$

The points of the contour calculated by this procedure are compared the ones obtained by the singularity method. This can be shown in Fig.4.

5. SUMMARY

A new mathematical approach is demonstrated for determination of the contour of the heated region developed in the adjacent rock mass around a geothermal well. The two-dimensional method of thermal singularities is expanded for a three-dimensional axisymmetric heat conduction problem. Instead of the solution of the differential equation of the heat conduction, two basic equations are written for the heat flux vector. Its components can be obtained from a thermal potential by simple derivation. The equpotential surfaces of this potential I.e. the isotherms can be determined without integration. The orthogonal set of surfaces form a stream function tangenting the heat flux vectors. These stream surfaces are adiabatic surfaces. There is no heat flux across them. The contour of the heated region such an adiabatic stream surface can be obtained as an axisymmetric surface of rotation. The validity of the model must of course checked by experiments. The points of the contour had been calculated from the measured temperature distribution of the flowing fluid through the well. Calculated and measured data are in rather good agreement.

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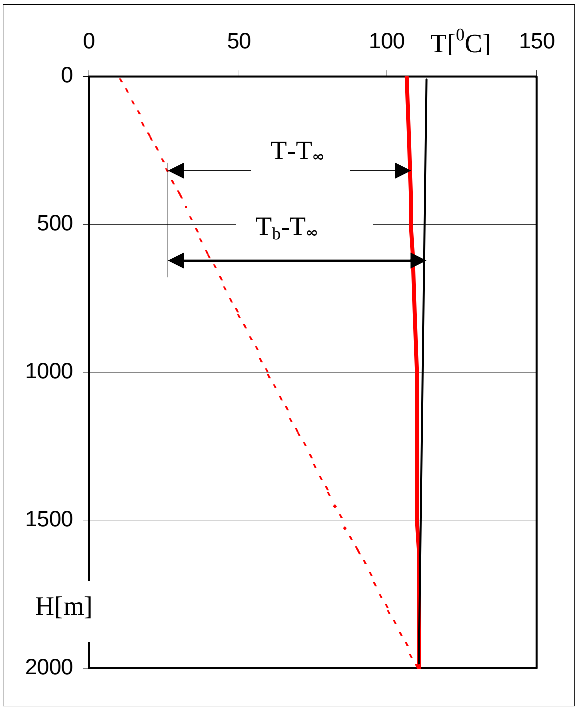

Fig.1. The temperature difference actuating the radial heat flux

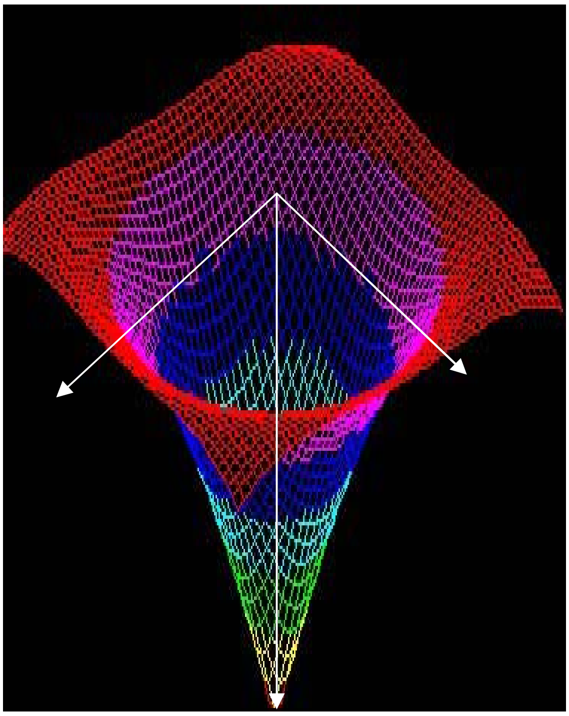

Fig.2. The contour of the heated region

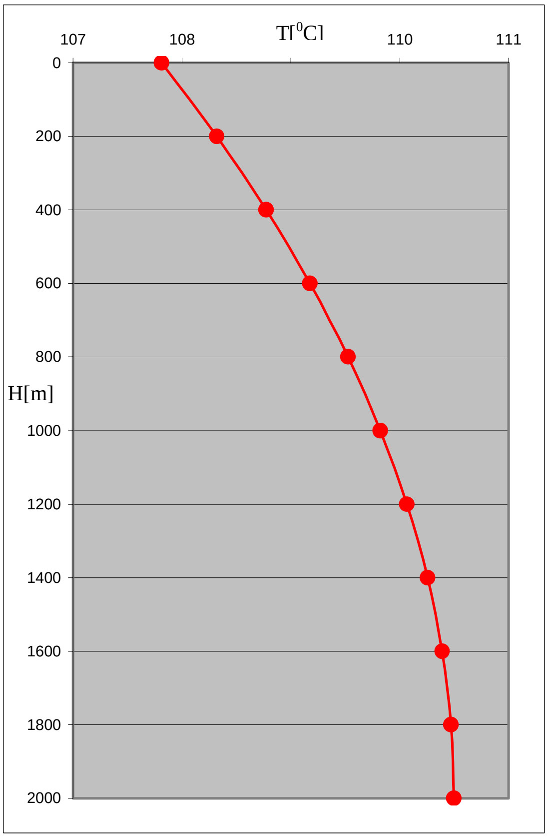

Fig.3. Measured temperature distribution

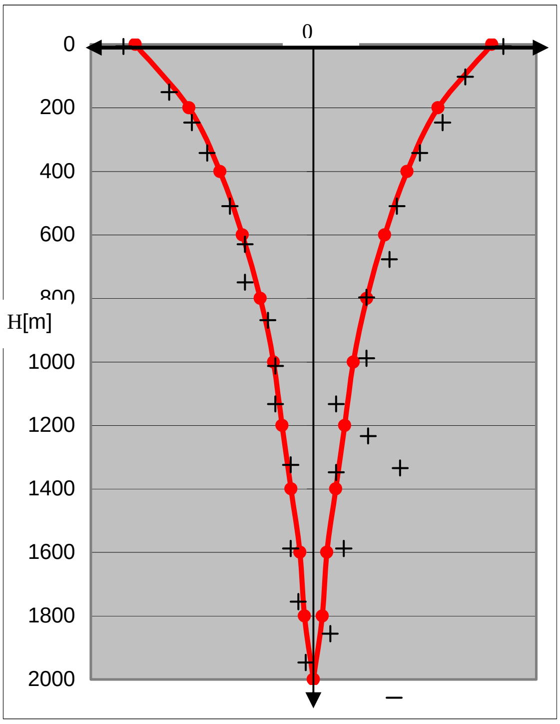

Fig.4. Comparison of the calculated contour to experimental data