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A Model for Dispersal of Eruption Ejecta

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ABSTRACT

Solid and liquid particulate materials that are ejected by hydrothermal or volcanic eruptions are subsequently dispersed by atmospheric wind currents while falling under gravity to the Earth's surface. Particle sizes are not uniform and may indeed change during flight owing to coalescence and/or fragmentation. Wind conditions (speed, direction and turbulence) may also change with elevation (and with time).

In an attempt to determine the most important physical factors, a quantitative model that reflects the above influences on particle dispersal is outlined. Analytic solutions to the mathematical model are sought wherever possible, since an analysis of the sensitivity of the predicted distributions to the numerous parameters involved is then more readily undertaken. Numerical simulations have also been made. Some example calculation results are compared with measured eruption deposit patterns published in the geological literature; these comparisons indicate that the proposed model reflects various characteristics of the measured data well.

1. INTRODUCTION

The deposit of ash from volcanic eruptions, pollen distribution by the wind, seabed contamination by dumping, and environmental pollution through airborne contaminants (solid or gaseous) are all able to be described by physical and mathematical models which combine advection and dispersion.

Measurements of neutrally-buoyant components as time-dependent concentrations, or of heavier-than-fluid components as spatially-distributed deposits, may be used to deduce the origin and release rates or total releases. These measurements may also give information about fluid flow conditions throughout the transport processes.

Recent work (Kathirgamanathan *et al.*, 2001, 2003, 2004) has been done on devising methods that use inverse models for gas releases into the atmosphere. Data from real-time concentration measurements made at several downstream locations on the ground can be used to deduce the location and release rate of a pollutant. Such information may allow calculation of total release and the prediction of atmospheric concentration as time continues to pass.

At present, the accuracy of the predictions are restricted by simplifying assumptions about the wind velocity and dispersion coefficients which represent the air turbulence. In practice, the precision will also be constrained by the inherent variability between similar releases due to turbulent dispersion.

The mathematical methods required, even for such simplified models, include both linear and non-linear optimization processes as well as some regularization techniques. However, comparison of calculated results with field data shows some success with the method (Kathirgamanathan *et al*., 2004).

This paper deals with heavier-than-fluid releases. It includes discussion of sample results calculated from an advection-dispersion model which takes account of lateral drift caused by the wind, settling of the released solid particles and their dispersion by air turbulence.

Previous versions of the model (McKibbin, 2003) have been enhanced: account is now taken of variation with elevation of wind speed and direction, particle settling speed and turbulent dispersion through a model atmosphere comprising layers. The description that follows is set in the context of a specific example, that of volcanically-erupted ash (small rock particles), but the analysis is the same for various other air-borne or water-transported particles.

Needless to say, the actual physical processes for distribution of eruption material are very complicated. Some of the main features captured by the model are as follows:

• The atmosphere is modelled as a layered system; within each layer, the wind is uniform in speed and direction, the settling speed for any given particle is constant and the turbulence length scales are uniform.

• The ground or bed surface is approximately horizontal – it is assumed that the fluid flow is parallel to the surface and that variation of topography is not severe enough to influence the average transport mechanisms.

• The eruption material is ejected to a certain height, where it is released into the wind – each particle quickly takes up a velocity which corresponds to the wind speed laterally and the particle's terminal speed (the "settling speed") vertically downwards; a similar fast transition occurs as particles pass from one layer to another.

• The turbulence within the air flow is modelled as having a certain characteristic length – since turbulence has a variety of scales, the length is a typical mean value for the flow.

2. AN ADVECTION–DISPERSION MODEL

A Cartesian coordinate system (x, y, z) is used, with the coordinates (x, y) measuring position on the ground relative to a fixed origin, and with z measuring vertical height above the ground surface. A cohort of particles of a certain size, with a total mass Q, is supposed to be released at time $t = 0$ at a point (x_0, y_0, H) which is at a height H above the point (x_0, y_0) on the ground. The particles are blown by a wind $\mathbf{u}(z) = (U(z), V(z), 0)$ with speed $W(z) = \sqrt{U^2 + V^2}$, while falling (settling) in the negative z-direction; they are simultaneously dispersed by the turbulent motion of the atmosphere.

If the concentration of the particles (mass per unit volume of air) at time $t > 0$ is denoted by $C(x, y, z, t)$, then mass conservation requires that

$$\frac{\partial C}{\partial t} = -\nabla \cdot \mathbf{q} \qquad (1)$$

where $\mathbf{q}$ is the solid particle mass flux per unit area, given by

$$\mathbf{q} = C\,\mathbf{u} - \underline{\mathbf{D}} \otimes \nabla C - C\,S\,\mathbf{k} \qquad (2)$$

Here, the first term $C\mathbf{u}$ represents the mass advection by the wind, $\underline{\mathbf{D}}(z)$ is a dispersion tensor and the last term is the flux due to falling under gravity, with $S(z)$ being the settling speed, or terminal speed, of a particle of the given size. The vector $\mathbf{k}$ is a unit vector in the positive z-direction (upwards).

The settling speed S of a particle which has density ρ_r and radius R falling in air with density ρ_a may be calculated from a balance of weight and drag forces in still air, given approximately by the correlation for spherical particles which are much denser than the fluid ($\rho_r >> \rho_a$):

$$\rho_r \left(\tfrac{4}{3} \pi R^3 \right) g = \tfrac{1}{2} C_S \rho_a \left(\pi R^2 \right) S^2 \qquad (3)$$

where g is the gravitational constant and C_S is the drag constant (Perry *et al.*, 1984). This gives the settling speed:

$$S = \sqrt{\frac{8}{3} \frac{\rho_r}{\rho_a} \frac{g}{C_s} R} \qquad (4)$$

which is proportional to the square root of the particle radius. The dispersion tensor is assumed to be of the form $\underline{\mathbf{D}}(z) = |\mathbf{u}|\,\underline{\mathbf{L}}$ where $\underline{\mathbf{L}}(z)$ is a dispersion mixing-length tensor reflecting the mean size of turbulence. Then, from Equation (2), the components in the (x, y, z) directions of the specific mass flux $\mathbf{q} = (q_x, q_y, q_z)$ are

$$\mathbf{q} = \left(UC - D_1 \frac{\partial C}{\partial x}, VC - D_2 \frac{\partial C}{\partial y}, -SC - D_3 \frac{\partial C}{\partial z} \right) \qquad (5)$$

Attainment of settling speed

One issue that is of concern is the assumption that the particles have attained terminal velocity. This is embodied in the assumption that the average particle velocity (the terminal velocity) is the resultant of that of wind $\mathbf{u}$ and that of the settling velocity $-S\mathbf{k}$ [where S is given by Equation (4)]. A particle released from rest or projected into the air will be accelerated by the combination of air resistance and gravity. We need to be assured that the particle reaches terminal velocity within a time and distance that are both small compared with typical large-scale values.

Some calculations reveal that, typically, rock particles with a diameter of 2 mm reach 99% of their settling speed ($S \approx 9$ m/s) within a time of 3 seconds while falling a distance of less than 20 m. For particles with a diameter of 2 cm, the time is 8 seconds and distance is 160 m, while for a diameter of 20 cm, the time and distance are 25 seconds and 1600 m.

These calculations show that for eruptions where particles are ejected several thousands of metres in to the air, the particles of size no greater than 2 cm diameter reach settling speed within a relatively short distance after release. The assumption made in the current model for smaller-sized particles therefore appears justified, but would not apply to very large rock particles.

3. UNIFORM ATMOSPHERE

If it is assumed that U and V are constants, i.e. the mean wind does not vary significantly with height, then, with each of the D_i constant, substitution of the expression for $\mathbf{q}$ from Equation (5) into (1) gives

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} - S\frac{\partial C}{\partial z} = D_1\frac{\partial^2 C}{\partial x^2} + D_2\frac{\partial^2 C}{\partial y^2} + D_3\frac{\partial^2 C}{\partial z^2}$$
$$+\ Q\delta(t)\delta(x-x_0)\delta(y-y_0)\delta(z-H) \qquad (6)$$

This advection-dispersion equation incorporates the initial condition, represented in terms of the Dirac delta function, of zero concentration everywhere at $t = 0$ except at the point (x_0, y_0, H), where a mass Q of particles is released. The far-field boundary conditions are of the form:

$$C \to 0 \quad \text{as} \quad x, y \to \pm\infty,\ z \to +\infty \qquad (7)$$

On the ground surface, $z = 0$, the vertical diffusive mass flux must be zero, which gives

$$\frac{\partial C}{\partial z} = 0 \quad \text{on} \quad z = 0 \qquad (8)$$

and the vertical mass flux onto the ground is

$$-q_z(x, y, 0, t) = S\,C(x, y, 0, t)$$

An alternative boundary condition instead of (8) is

$$C \to 0 \quad \text{as} \quad z \to -\infty \qquad (8)'$$

which, while not ensuring that the diffusive flux at $z = 0$ is exactly zero, does not greatly change the overall dispersion pattern and allows a simpler solution of the advection-diffusion equation (6).

Comparison of the solutions found using the two different boundary conditions (8) and (8)' indicates that the total mass deposited on the ground is the same (= Q) in both cases, with only a small difference in the predicted lateral distribution. The simpler solution using the boundary conditions (7) and (8)' will be presented here. [The solution using boundary conditions (7) and (8) is also available, but is mathematically more complicated to write down.]

Solution

The analytic solution to Equation (6) with boundary conditions (7) and (8)' is (Kevorkian, 1993; Logan, 1998):

$C(x, y, z, t) =$

$$\frac{A}{t^{3/2}} exp\left[-\frac{(x-x_0-Ut)^2}{4D_1 t} - \frac{(y-y_0-Vt)^2}{4D_2 t} - \frac{(z-H+St)^2}{4D_3 t} \right] \qquad (9)$$

where

$$A = \frac{Q}{8\pi^{3/2}\sqrt{D_1 D_2 D_3}} \qquad (10)$$

The downward flux at the ground ($z = 0$) is, for boundary condition (8)',

$$-q_z(x, y, 0, t) = \left[SC + D_3 \frac{\partial C}{\partial z} \right]_{z=0}$$

$$= \frac{A}{2} \left\{ \frac{H}{t^{5/2}} + \frac{S}{t^{3/2}} \right\} \times$$

$$\times \; exp\left[-\frac{(x - x_0 - Ut)^2}{4D_1 t} - \frac{(y - y_0 - Vt)^2}{4D_2 t} - \frac{(-H + St)^2}{4D_3 t} \right] \quad (11)$$

which gives the mass deposition rate per unit area, assuming no bouncing or rolling. The total deposition (mass per unit area) $f(x, y)$ at point (x, y) on the ground is given by

$$f(x, y) = \int_0^\infty -q_z(x, y, 0, t) \; dt \quad (12)$$

which may be found, after some calculation, to be

$$f(x, y) = \frac{Q}{32\pi\sqrt{D_1 D_2 D_3}} \; \frac{(2\alpha\beta + 1)H + 2\alpha^2 S}{\alpha^3} \times$$

$$\times \; exp\left[\frac{U(x - x_0)}{2D_1} + \frac{V(y - y_0)}{2D_2} + \frac{HS}{2D_3} - 2\alpha\beta \right] \quad (13)$$

where $\alpha(x, y)$ and β are parameters, both positive and defined by

$$\alpha = \frac{1}{2}\left[\frac{(x - x_0)^2}{D_1} + \frac{(y - y_0)^2}{D_2} + \frac{H^2}{D_3} \right]^{\frac{1}{2}}$$

$$\beta = \frac{1}{2}\left[\frac{U^2}{D_1} + \frac{V^2}{D_2} + \frac{S^2}{D_3} \right]^{\frac{1}{2}}$$

An example calculation

To provide an example calculation, the data in Table 1 are used.

Here 1000 tonnes of rock particles of diameter 2 mm are assumed to be released at an altitude of 5 kilometres above the origin into a 20 m/s (72 km/hr) wind with a mean turbulence length scale of 100 m. The settling speed S of the particles is calculated from Equation (4) to be approximately 8.9 m/s.

Figure 1 shows the 10% contours of the ground distribution of the fallout. The point where particles which were not subject to dispersion would land is marked by a star (∗), a distance x^* from the point (o) directly beneath the release point. The time taken to fall to the ground for such a particle is

$$t^* = \frac{x^*}{U} = \frac{H}{S} \approx 9.4 \text{ minutes}$$

and

$$x^* = \frac{U}{S} H = \frac{20}{8.9} \times 5{,}000 \approx 11{,}200 \text{ m}$$

Table 1: Data for example calculation of fallout distribution from a release of particles.

parameter	*SI units*	*value*
Q	tonne = 10^3 kg	1 000
ρ_r	kg/m^3	1 500
ρ_a	kg/m^3	1.3
C_S	–	0.38
g	m/s^2	9.8
U	m/s	20
V	m/s	0
$W = \lvert\mathbf{u}\rvert$	m/s	20
L_1	m	100
L_2	m	100
L_3	m	100
$D_1 = \lvert\mathbf{u}\rvert L_1$	m^2/s	2000
$D_2 = \lvert\mathbf{u}\rvert L_2$	m^2/s	2000
$D_3 = \lvert\mathbf{u}\rvert L_3$	m^2/s	2000
x_0	m	0
y_0	m	0
H	m	5 000
R	m	0.001
S [calculated]	m/s	8.9

As can be seen from Figure 1, the maximum fallout concentration is at a point on $y = 0$ where $x < x^*$. This is a characteristic of the dispersion mechanism, which moves some of the particles at a faster rate towards the ground, as well as some upwards, sideways, etc. The longer tail downwind is due to those particles that move higher into the air and are carried further before falling on the ground.

Such a profile of historic ashfall deposits is common (see examples in Sparks, 1997), and reflects a uniform wind pattern for the duration of an eruption. Generally, of course, the mass is "emplaced" in the atmosphere at different heights, and the particles are of various sizes.

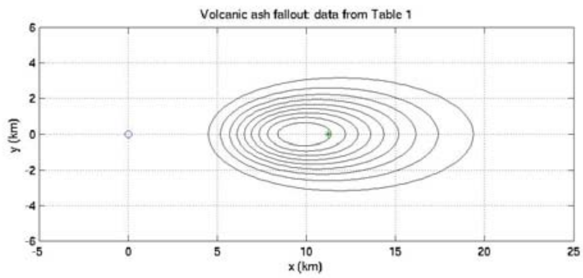

Figure 1: 10% contours of mass distribution on the ground after the release of particles; the various parameters are given in Table 1.

Particle size distributions

In general, releases are composed of a range of different-sized particles. It is clear from the above calculation that the fallout distribution of particles with large settling speeds will be closer to the release point. So, the overall particle size distribution at a point on the ground depends on its position. Points far from the source release will have a predominance of smaller lighter particles, while the regions near the release point will be rich in larger particles. An example calculation was demonstrated in McKibbin (2003).

The distribution concentrations are measured in kg/m^2 in SI units. This mass distribution may also be interpreted as a "depth of deposit" if information on the packing of the material after deposit is known. In general, the deposit has some average porosity ϕ, which reduces the effective volumetric density of the rock material (intrinsic density ρ_r) to a smaller value $(1 - \phi)\,\rho_r$.

Particle agglomeration

Such a model as outlined above should also be able to take account of agglomeration of particles within the atmosphere as they move along. If atmospheric conditions within some zone produce condensation within humid air, the resulting water droplets may contact particles that then coalesce, linked by so-called "liquid bridges". Further agglomeration may then occur between these wet clusters. This increases the mass of a "particle" which is now a cluster of smaller particles. The average particle cross-sectional area increases and Equation (4) shows that the settling speed increases.

The next section demonstrates how the model is extended to take account of such a phenomenon occurring in one region of the atmosphere, and the consequential change in fallout distribution on the ground.

4. A LAYERED ATMOSPHERE MODEL

To deal with the phenomena mentioned above – variation of the wind velocity and turbulence characteristics with elevation, and also the agglomeration (and possible fragmentation) of particles during their flight – the model is structured as a horizontally-layered system. The atmosphere is modelled as a system of layers, within each of which the wind velocity, turbulence length scales and the particle settling speeds are uniformly constant. The number of layers is decided based on the degree of accuracy required to match the actual measured wind profile.

In this work, the layers are numbered from top (Layer 1) to bottom (Layer N). To simplify matters a little for the illustrations given here, it is supposed that the particles are released into the topmost layer (Layer 1). Within Layer j, which lies in the region $Z_j < z < Z_{j-1}$, the mass concentration C satisfies Equation (6), but with the parameters appropriate to that layer:

$$\frac{\partial C}{\partial t} + U_j \frac{\partial C}{\partial x} + V_j \frac{\partial C}{\partial y} - S_j \frac{\partial C}{\partial z} = D_{1j} \frac{\partial^2 C}{\partial x^2} + D_{2j} \frac{\partial^2 C}{\partial y^2} + D_{3j} \frac{\partial^2 C}{\partial z^2} + Q\delta(t)\delta(x-x_0)\delta(y-y_0)\delta(z-H) \quad (14)$$

Since the point (x_0, y_0, H) is contained in Layer 1, the last (source) term has effect only in that layer. At the horizontal layer boundaries, there is a requirement that the mass density (concentration) of the solid particles and the downwards mass flux be continuous. Otherwise, boundary conditions similar to those given in (7) and (8) apply. The result is a system of N partial differential equations with the appropriate corresponding initial and boundary conditions; the problem is mathematically well-posed.

To further simplify the model, it is supposed that the vertical component of turbulent dispersion is negligible. This has been noted by several authors [e.g. Carey (1996) states that above an elevation of 500 m the vertical diffusion coefficient is approximately zero]. Putting D_{3j} = 0 in the N Equations (14) reduces each to first-order in z, and thereby reduces the number of boundary conditions needed in that direction. The requirement for continuity of C is therefore dropped.

The resulting distribution of a mass released at $t = 0$ may be conceived as a horizontal "sheet" of particles which is falling at the local settling speed while moving with the local wind and dispersing with the local turbulent activity. The particles all arrive at a layer interface simultaneously, and then fall through to the next layer, and so on until they reach the ground.

The solution is formally given by:

$$C(x, y, z, t) = \frac{Q}{4\pi\sqrt{D_{1w}D_{2w}}} exp\left[-\frac{(x-X_w)^2}{4D_{1w}} - \frac{(y-Y_w)^2}{4D_{2w}}\right]\delta(z-Z_w) \quad (15)$$

for $t_{j-1} < t < t_j$ where t_j is the mean time taken for a particle to fall to the layer interface at Z_j; these times are given by

$$t_0 = 0, \quad t_1 = \frac{H-Z_1}{S_1}, \quad t_j = t_{j-1} + \frac{Z_{j-1}-Z_j}{S_j} \ \text{for}\ j \ge 2.$$

The other parameters in Equation (15) are all functions of time, derived from the separate layer parameters that are weighted by the time spent in traversing them. They are the mean horizontal displacement components $X_w(t)$, $Y_w(t)$; the downwards vertical displacement $Z_w(t)$; and the weighted dispersion coefficients $D_{1w}(t)$ and $D_{2w}(t)$ in the x- and y-directions respectively. For $t_{j-1} < t < t_j$, when the particles are in Layer j (j = 1, …, N) these parameters are given by:

$$\begin{aligned}
X_w &= x_0 + U_1(t_1-t_0) + U_2(t_2-t_1) + \ldots + U_j(t-t_{j-1}) \\
Y_w &= y_0 + V_1(t_1-t_0) + V_2(t_2-t_1) + \ldots + V_j(t-t_{j-1}) \\
Z_w &= H - S_1(t_1-t_0) - S_2(t_2-t_1) - \ldots - S_j(t-t_{j-1}) \\
D_{1w} &= D_{11}(t_1-t_0) + D_{12}(t_2-t_1) + \ldots + D_{1j}(t-t_{j-1}) \\
D_{2w} &= D_{21}(t_1-t_0) + D_{22}(t_2-t_1) + \ldots + D_{2j}(t-t_{j-1})
\end{aligned}$$

The distributions of such a release when it arrives at the layer interfaces and the ground are shown in Figure 2. Here a mass Q = 1000 tonnes of uniformly-sized particles is released 2000 m above the ground. The interfaces at 1200 and 500 m above the surface separate Layers 1 and 2, and Layers 2 and 3 respectively. The wind velocity vectors in Layers 1, 2 and 3 are (2, 1), (–2, 4) and (–10, –8) m/s in the (x, y) directions respectively, and the particle settling speed is S = 3 m/s. The horizontal turbulence length scales are all 100 m. The wind vectors are shown as arrows within each layer. Note that these parameters are deliberately not taken from actual measured data, but are used for illustration only.

It can be seen how the particle mass concentration flattens out as it spreads, and how the different wind directions move the particles in different directions. (Note that the vertical scale of the distributions is exaggerated for clarity.)

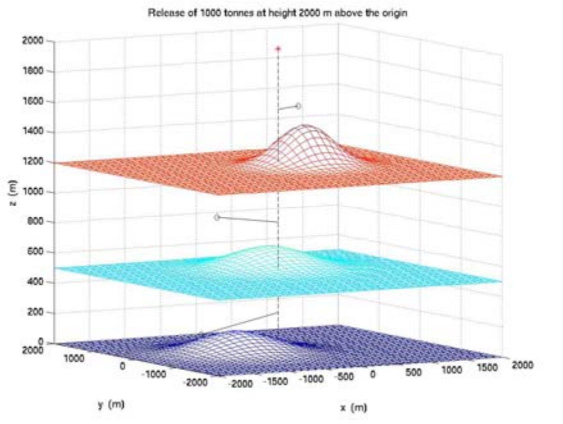

Figure 2. Distributions at layer interfaces and the surface deposit of a mass of particles released 2000 m above the ground – the vertical scale of the concentration profiles is exaggerated.

Agglomeration of particles while in flight

The above analysis is able to be modified to allow for the situation where particles in a particular region of the atmosphere undergo a change in settling speed due to water condensation or adherence during cooling, which cause agglomeration. Mathematically, a particular section of particles is removed from the distribution at a layer interface, and added back with a different settling speed. The analysis requires some mathematical integration and the use of the original solution given by Equation (15). For brevity, the mathematical details are not given here, but an illustrative example using calculations from the analytical solution is shown in Figure 3.

Here a mass Q = 1000 tonnes of uniformly-sized particles is released 2000 m above the ground. An interface is at 1200 m above the surface. The wind velocity is in the *x*-direction with speeds in Layers 1 and 2 being 2 and 6 m/s respectively. The settling speed of the particles when released is 3 m/s. The horizontal turbulent length scales are 200 and 20 m in the downwind and cross-wind directions respectively.

At the interface between Layers 1 and 2, the particles which arrive from above into the rectangular region described by $100 \le x \le 300$ m, $-100 \le y \le 100$ m have their settling speeds increased to 23 m/s. The remaining particles continue falling and dispersing with the same settling speed as in Layer 1. The resulting mass distributions at the interface between Layers 1 and 2 and on the ground are shown in Figure 3(a). Note that now the ground deposit contours contain two maxima; one is due to the bulk of the particles, while the maximum closer to the release point is due to the faster-falling cohort. A contour plot of the ground deposit in Figure 3(b) demonstrates this more clearly.

This example shows the characteristics of ashfall contours from the Mount St Helens eruption of 18 May 1980 (see, for example, Figure 1 of Carey, 1996), which show two deposit thickness maxima. While such a deposit profile might easily be described by a model which has two distinct particles sizes in the release, the above analysis shows how agglomeration after release of a cohort of uniformly-sized particles can produce the same effect.

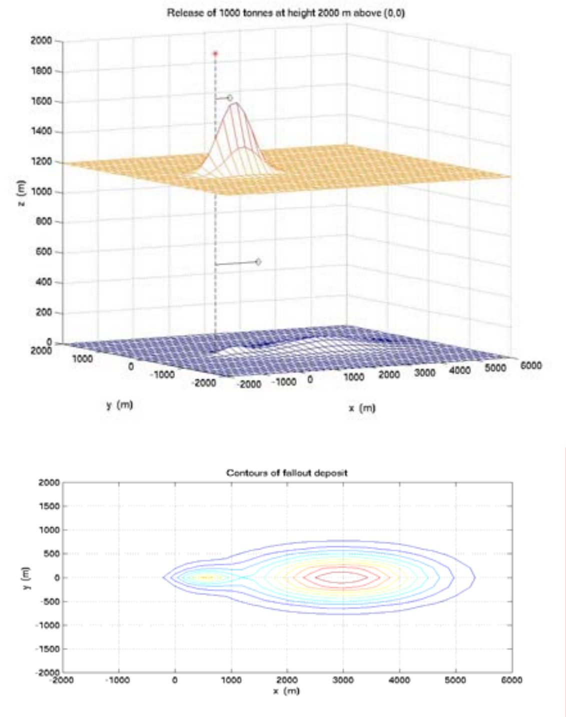

Figure 3: Initial release of particles modified by agglomeration in the rectangular region $100 \le x \le 300$ m, $-100 \le y \le 100$ m at the layer interface. See text for discussion.
(a) Distribution at layer interface and of ground deposit.
(b) Contour plot of ground deposit.

Because the current model is analytic in form, simple adjustment of parameters could allow more complicated deposit profiles to be modelled without repetition of time-consuming full numerical simulations.

5. CONCLUSION

Some example calculations of the distribution of deposits of solid particles released into a fluid flow have been made using a simplified advection-dispersion model. The method treats the system as a sequence of layers, with flow parameters uniform within each layer but varying from layer to layer. Thereby, a non-uniform flow profile may be approximated, as well as allowing for particle characteristics to change as they pass from one layer to another.

The model is able to take into account both the variation in particle sizes within an eruption release and the change in particle size due to agglomeration or fragmentation in particular zones in the atmosphere during the flight. The main distribution mechanisms of wind-induced drift, gravitational settling and turbulent dispersion are included.

It should also be noted that the method can also be used for cases where the particles fall through the atmosphere and then into water, or when the release is made directly into water. Settling speeds in fluids other than air are readily calculated using Equation (4) with a different fluid density.

Extensions to the model are currently being made, especially with respect to calculating releases made at different heights, and at different times, and where the discharge is continuous.

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