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A. Lehikoinen[1]
Department of Physics, University of Kuopio, Finland

S. Finsterle[2]
Earth Sciences Division, Lawrence Berkeley National Laboratory, California, USA

A. Voutilainen
Department of Physics, University of Kuopio, Finland

M.B. Kowalsky
Earth Sciences Division, Lawrence Berkeley National Laboratory, California, USA

J.P. Kaipio
Department of Physics, University of Kuopio, Finland

[1]Department of Physics, University of Kuopio, P.O.Box 1627, FIN-70211, Kuopio, Finland.
Ph. +358 17 16258 Fax +358 17 162373

[2]Earth Sciences Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, MS 90-1116, Berkeley, California 94720, USA.
Ph. +1510-487-6455 Fax +1510-486-5686

A BAYESIAN NONSTATIONARY INVERSION APPROACH FOR IMAGING FLUID FLOW

A. LEHIKOINEN[1], S. FINSTERLE[2], A. VOUTILAINEN[1], M.B. KOWALSKY[2], J.P. KAIPIO[1]

[1]Department of Physics, University of Kuopio, Kuopio, Finland
[2]Earth Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, USA

SUMMARY We present a new methodology for imaging the evolution of electrically conductive fluids in porous media. The inversion problem is formulated as a state estimation problem. The approach is based on an evolution-observation model and is solved using an extended Kalman filter algorithm. The example we consider involves the imaging of time-varying distributions of water saturation in porous media using time-lapse electrical resistance tomography (ERT). The complete electrode model (with Archie's law relating saturations to electrical conductivity) is used as the observation model. The evolution model we employ is a simplified (approximate) model for simulating flow through partially saturated porous media. We propose to account for approximation errors in the evolution model by constructing a statistical model of the differences between the accurate and approximate representations of fluid flow, and by including this information in the calculation of the posterior probability density of the estimated system state. The proposed method provides improved estimates of water saturation distribution relative to traditional reconstruction schemes that rely on conventional stabilization methods (e.g., using a smoothness prior) and relative to the extended Kalman filter without incorporating the approximation error method. Finally, the approximation error method allows for the use of a simplified and computationally efficient evolution model in the state estimation scheme. The methodology presented here for unsaturated flow through porous media may be extended for applications of nonisothermal multiphase flow in fractured geothermal reservoirs using a variety of geophysical methods.

1. INTRODUCTION

Monitoring soil water content and its change with time is of great importance for understanding flow and transport processes in variably saturated porous media. The use of geophysical methods for inferring spatial and temporal changes in water content has proven useful, see for example Binley *et al.* (2002). For example, the sensitivity of the electrical conductivity to water content enables the use of electrical resistance tomography (ERT) to monitor the water content distribution.

Electrical resistance tomography has been applied to a wide range of environmental and hydrogeological investigations, including the monitoring of tracer movement in the saturated zone (Kemna *et al.*, 2002). ERT has also been used in monitoring spatial and temporal variations of soil water content in the vadose zone (Zhou, 2001). For example, cross-borehole electrical resistivity tomography has been used to image the resistivity distribution of the vadose zone before and during infiltration experiments (Daily, 1992).

In ERT, electric currents are injected into the target through electrodes placed in boreholes, and the resulting voltages are measured using the same electrodes. The internal conductivity distribution is estimated based on the current and voltage data.

Classical tomographic reconstruction approaches analyze geophysical data from a single measurement frame only, where each frame may consist of voltage measurements corresponding to several current injections. Because a frame refers to a single reconstruction, the associated data are assumed to be obtained from a time-invariant target. Multiple reconstructions from frames at different times are then combined to visualize time-varying hydrological phenomena. In the stationary reconstruction approach the errors between sequential reconstructions tend to be large and typically unrealistic because the approach do not incorporate the temporal prior knowledge of the target into the reconstruction.

Recent results (Seppänen, 2000; Kaipio, 2004) indicate that by combining information from ERT measurements and an appropriate evolution model, the accuracy of the estimates may be improved significantly in the case of time-varying targets. The basic idea behind the nonstationary inversion approach is to incorporate models for the temporal behavior of the target into data processing. The use of the models for the time evolution of the water content distribution can be interpreted as temporal prior models for the system. Although the rate of change of hydrological processes in the subsurface is relatively slow, the use of the non-stationary approach stabilizes the reconstruction problem.

In this paper we develop a nonstationary inversion approach to monitor moisture flow in unsaturated porous media. We consider the problem as a state estimation problem. In order to evaluate the performance of the proposed approach in comparison with the traditional Tikhonov regularization method, a numerical test case is presented in which we monitor the water saturation distribution as water is injected into an unsaturated porous medium.

2. METHODS

2.1 Observation Model

Electrical resistance tomography (ERT) is an imaging method in which conducting targets are monitored via electrical measurements. In ERT, electric currents I are injected into the target through electrodes, and the resulting voltages V between electrode pairs are measured. The forward model is the computational model which predicts the voltage measurements given the electrical conductivity distribution $\sigma = \sigma(x)$ of the monitored domain Ω_{ert} and the injected current pattern $I = \{I_1, ..., I_L\}$, where L denotes the number of electrodes. The boundary of the domain Ω_{ert} is denoted $\partial\Omega_{ert}$. The most accurate forward model for ERT is the complete electrode model (Cheng, 1989), defined by the following boundary value problem

$$\nabla \cdot (\sigma \nabla u) = 0, \qquad x \in \Omega_{ert} \tag{1}$$

$$u + z_l \sigma \frac{\partial u}{\partial \bar{n}} = U_l, \qquad x \in e_l, l = 1, ..., L \tag{2}$$

$$\int_{e_l} \sigma \frac{\partial u}{\partial \bar{n}} dA = I_l, \qquad l = 1, ..., L \tag{3}$$

$$\sigma \frac{\partial u}{\partial \bar{n}} = 0, \qquad x \in \partial\Omega_{ert} \tag{4}$$

where $u = u(x)$ is the electrical potential, e_l is the l^{th} electrode, z_l is the contact impedance between the l^{th} electrode and the domain Ω_{ert}, U_l is the potential on the l^{th} electrode, I_l is the injected current and $\bar{n}$ is the outward unit normal vector. In addition, the charge conservation law needs to be fulfilled, and the potential reference needs to be fixed. Thus, the following conditions are written

$$\sum_{l=1}^{L} I_l = 0, \sum_{l=1}^{L} U_l = 0. \tag{5}$$

The Finite Element approximation of the forward problem can be written in the form (Vauhkonen, 1997)

$$U(\sigma) = R(\sigma) I, \tag{6}$$

where $U(\sigma) \in \mathbf{R}^{L \times N_I}$ is a matrix of measured voltages between the electrodes, $R(\sigma) \in \mathbf{R}^{N_v \times L}$ is a resistivity matrix, and $I \in \mathbf{R}^{L \times N_I}$ is the matrix of current patterns. Here, N_I is the number of injected current patterns.

In this paper, we assume that Archie's law (Archie, 1942) accurately relates water saturation to electrical conductivity. Archie's law is described as

$$\sigma(S) = \sigma_w \phi^{ce} S^n, \tag{7}$$

where σ_w is the conductivity of the aqueous phase, ce is called the cementation index, and n is the saturation index.

The inverse problem of ERT is written in terms of water saturation $S \in \mathbf{R}^N$. By substituting (7) into (6), the forward model can be written as

$$V = U^*(S), \tag{8}$$

where the mapping $U^* : \mathbf{R}^N \mapsto \mathbf{R}^{N_V}$ is $U^*(S) = U(\sigma(S))$, and N_v is the number of voltage observations. We assume that measurements are corrupted by additive Gaussian noise ε with zero mean and covariance Γ_ε. This is the commonly used measurement error model for ERT and the statistics of the measurement noise can be measured. With the following assumptions the observation model (8) can be written in the form

$$V = U^*(S) + \varepsilon. \tag{9}$$

The reconstruction of the water saturation distribution requires multiple measurements corresponding to several current injections. One ERT frame is defined as the voltage data corresponding to N_I current injection patterns injected during a short time period in which the target distribution can be considered time-invariant. The fact that hydrological processes are often relatively slow justifies the assumption that the saturation distribution $S_t \in \mathbf{R}^N$ does not change significantly while the data for a specific frame are collected. Thus, by stacking the models (9) corresponding to all N_I current injections, the observation model at time t can be written as

$$V_t = U^*(S_t) + \varepsilon_t \tag{10}$$

Here, $V_t \in \mathbf{R}^{N_v N_I}$ is the vector containing the voltage measurements corresponding to current patterns applied at time t. We assume here that the forward problem is relatively accurate so that the approximation errors due to the finite element model for the forward problem are smaller than the additive noise. We also assume that the parameters of Archie's law are accurately known.

2.2 Evolution Model

The flow of water in variably saturated porous media is modelled with the Richards' equation (see Bear (1988) for a general description), which can be written as

$$\varphi \frac{\partial S}{\partial t} + \nabla \left[\frac{K(S)}{\rho_w g} \nabla P_c(S) - K(S) \bar{z} \right] = 0, \tag{11}$$

where φ is the porosity, S is the water saturation, K is the unsaturated hydraulic conductivity and P_c is capillary pressure (both nonlinear functions of water saturation), ρ_w is the water density, g is the gravitational constant, and $\bar{z}$ is the unit vector, positive upward.

The hydraulic conductivity is given by

$$K = k\frac{k_{rel}(S)\rho_w g}{\mu_w}, \quad (12)$$

where k is the absolute permeability, k_{rel} is the relative permeability, which is a nonlinear function of water saturation, and μ_w is the dynamic viscosity of water.

Van Genuchten's parametric model (van Genuchten, 1980) for relative permeability k_{rel} and capillary pressure P_c can be written as

$$P_c = -\alpha^{-1}(S_e^{-1/m} - 1)^{1-m}, \quad (13)$$

$$k_{rel} = \sqrt{S_e}(1-(1-S_e^{m^{-1}})^m)^2, \quad (14)$$

$$S_e = \frac{S - S_{wr}}{1 - S_{wr}}, \quad (15)$$

where m and α are soil-specific parameters, S_e is the effective water saturation, S_{wr} is the residual saturation.

We assume that the relative permeability and capillary pressure functions are valid and the hydraulic parameters are known (however, our nonstationary inversion approach allows for uncertainty in hydraulic parameters to be considered, as will be discussed in the example given below). We specify the following initial and boundary conditions:

$$S(x,0) = S_0(x), \qquad x \in \Omega_d \quad (16)$$

$$\frac{\partial S}{\partial \overline{n}} = 0, \qquad x \in \partial\Omega_d \quad (17)$$

$$S(x,t) = S_{in}(x,t), \qquad x \in \partial\Omega_{in} \quad (18)$$

where $S_0(x)$ is the initial water saturation distribution, and $\overline{n}$ is the outward unit normal vector. Water saturation is kept constant at $S_{in}(x,t) = 1$ at the boundary denoted by $\partial\Omega_{in}$, representing a point release of water from a borehole located at x_{in}.

The numerical solution of equations (1)-(8) using the Finite Volume Method (FVM) yields the (forward) evolution model

$$S_{t+1} = F(S_{t+1}; k), \quad (19)$$

where S_t are the discretized saturation values at the nodes at a given time, and F is a nonlinear function of water saturation. This is the forward model for the evolution of water saturation; in this formulation, all hydraulic parameters are assumed to be known, although in our nonstationary inversion approach uncertainty in any parameter can be modelled, section 2.3.

For the simulated measurements used in the synthetic example, this model is used to compute the true time evolution of the saturation in a dense mesh. In this study the variably saturated flow simulator TOUGH2 (Pruess *et al.*, 1999) is used for this purpose.

The aim is to estimate the water saturation distribution as a function of space and time. We use Richards' equation as the evolution model in our nonstationary inversion approach. The Richards' equation is highly nonlinear and we assume that absolute permeability is not known exactly. Additional approximations (such as discretization errors) may lead to errors in the solution to the Richards equation, (Kaipio, 2004). Thus, the evolution model is written as

$$S_{t+1} = S_t + F(S_{t+1}) + \omega_t^1, \quad (20)$$

where ω_t^1 is a discrete time noise process representing modelling errors (Kaipio, 2004). Equation (20) is referred to as the state evolution model for water saturation.

2.3 Enhanced Error Model

Huttunen (2006) developed an approximation error method to handle uncertainties in nonstationary inverse problems. In this framework, the state noise term due to the uncertainties in the permeability is modeled as follows:

$$\begin{aligned} S_{t+1} &= S_t + F(k, S_{t+1}) + \omega^1{}_t \\ &= S_t + F(k^*, S_{t+1}) + (F(k, S_{t+1}) - F(k^*, S_{t+1})) + \omega^1{}_t \quad (21) \\ &= S_t + F(k^*, S_{t+1}) + \omega^1{}_t + \omega^2{}_t, \end{aligned}$$

where the best guess of the permeability (e.g., based on values reported in the literature for settings that are geologically similar to the ones being investigated) is k^*, which is used in inversion. Here, k is the realization of the heterogeneous permeability field considered to be reality. Thus, we use the approximation $F(k^*, S_{t+1})$ in the evolution model and interpret the term $(F(k, S_{t+1}) - F(k^*, S_{t+1}))$ as the state noise that is due to the uncertainty in the permeability.

2.4 Extended Kalman Filtering

State estimation refers to the problem of estimating the time-varying saturation distribution from ERT measurements, where the system is described using a state space representation. In hydrological applications, both the physical evolution and the petrophysical models are usually nonlinear. In this paper, we consider the additive noise model for the evolution and observation models so that we can write

$$V_t = g_t(S_t) + \varepsilon_t, \quad (22)$$

$$S_{t+1} = f_t(S_t) + \omega_t. \quad (23)$$

In our case, the state space representation consists of Eqs. (10) and (21). In nonlinear problems, suboptimal estimates can be obtained using an extended Kalman filter (EKF) algorithm (Anderson, 1979). The EKF recursions can be written as

$$S_{t|t-1} = f_{t-1}(S_{t-1|t-1}), \quad (24)$$

$$\Gamma_{t|t-1} = J_{f_{t-1}}\Gamma_{t-1|t-1}J^T{}_{f_{t-1}} + \Gamma_{\omega_{t-1}}, \quad (25)$$

$$K_t = \Gamma_{t|t-1}J^T{}_{g_t}(J_{g_t}\Gamma_{t|t-1}J^T{}_{g_t} + \Gamma_{\varepsilon_t})^{-1}, \quad (26)$$

$$\Gamma_{t|t} = (I - K_t J_{g_t})\Gamma_{t|t-1}, \quad (27)$$

$$S_{t|t} = S_{t|t-1} + K_t(V_t - g_t(S_{t|t-1})). \quad (28)$$

The computation of Jacobian J_{g_t} associated with the observation model is presented in Vauhkonen (2004). The Jacobian associated with the evolution model J_{f_t} is computed numerically using TOUGH2 (Pruess *et al.*, 1999).

3. RESULTS

We consider a simulated two-dimensional ERT survey collected during transient unsaturated flow induced by the injection of water from a point source into an initially dry heterogeneous porous medium. Electrodes are placed in two straight boreholes that are 5 meters apart. The electrodes are installed at 1.2 meters interval along the boreholes.

3.1 Simulation of Measurements

The true evolution of the water saturation S_t is computed by solving the Richards' equation with the TOUGH2 simulator. The only non-uniform flow parameter is the absolute permeability, and its log value is modelled as a random field. The log permeability field was generated using sequential Gaussian simulation (Deutsch and Journel, 1992). The parameters for this simulation are summarized in Table 1.

Table 1
Summary of parameters used to generate the water saturation data

Description	Parameter values
Unsaturated flow model parameters, Eqs. (11)-(12)	$\mu_w = 1.002 \times 10^{-3} Pas$ $\rho_w = 1000 kgm^{-3}$ $\phi = 0.3$
Capillary pressure and relative permeability, Eqs. (13)-(15)	$m = 0.628$ $\alpha = 0.001 Pa^{-1}$ $S_{wr} = 0.083$
Archie's law, Eq. (7)	$ce = 2$ $n = 2$ $\sigma_w = 0.01 Sm^{-1}$
Spherical semi-variogram model used to generate spatially correlated log permeability, as defined by Deutsch and Journel (1992)	a =0.4m (range) $c = 1.0$ (sill) Anisotropy factor= 5 Rotation angle=20^o (from horizontal)

The initial saturation was uniform and set at a value of 0.05 with the exception of the injection point, where water was supplied by keeping the saturation constant at value 0.99.

The voltages were simulated by using the numerical solution of the complete electrode model. The computed voltages were then corrupted with additive noise consisting of two components, both being Gaussian with zero mean. The standard deviation of the first component was 1% of the measurement, and the standard deviation of the second component was a 0.1% of the maximum voltage.

3.2 Numerical results

The meshes used in inversion were sparser and different from those used in the computation of the synthetic measurements. The model for the measurement noise in the inversion was also different from that in the synthetic measurements: it was zero mean but the covariance matrix is approximated as a time invariant diagonal matrix $\Gamma_{\varepsilon_t} = \sigma^2{}_v I$ with a standard deviation σ_v that was 0.1% of the difference between the maximum and minimum voltage difference.

Furthermore, we assumed that we knew all other associated parameters except the absolute permeability field. We assumed an isotropic, homogeneous permeability $k = 1 \times 10^{-12}$ in the evolution model.

To approximate the statistics of the state noise term ω_t^2, we generated 250 samples from the statistical model for k and computed the time-dependent Gaussian approximation for the statistics of the water saturation distribution. The random samples are shown in Figure 1. We assumed that we knew all the parameters in sequential Gaussian simulation (though the seed number was changed each time a random sample was drawn). Approximation errors are appropriately incorporated into the evolution model.

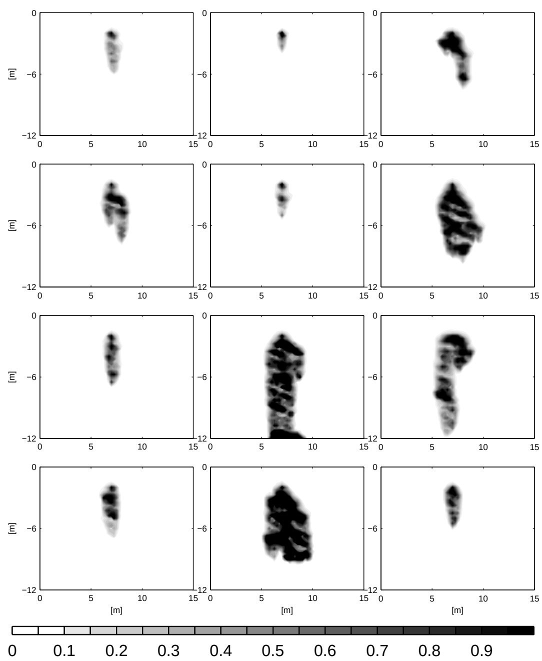

Figure 1. Simulated water distribution 90 hours after injection for 12 random permeability fields generated for the construction of a statistical enhanced error model.

For comparison, we also compute conventional stationary estimates of the water saturation based on the same simulated ERT data. The stationary estimates were calculated using an approach based on Tikhonov regularization (Kaipio, 2004). The water saturation distributions estimated with the proposed nonstationary and stationary approaches are presented in Figure 2. Comparing the simulated, true saturation distributions (left column) with the corresponding reconstructions shows clearly that the nonstationary estimates are better than the stationary estimates.

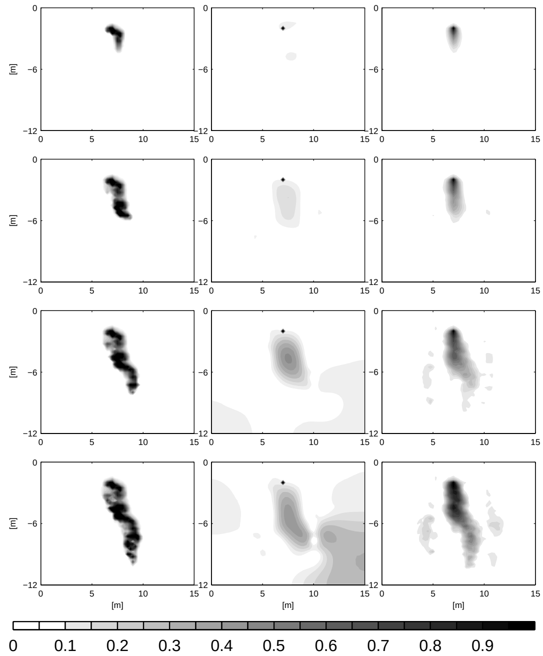

Figure 2. Left column: True, simulated saturation distributions; center column: stationary reconstruction; right column: nonstationary reconstruction.

4. CONCLUSION

In this study we proposed a nonstationary reconstruction approach for ERT data collected in the unsaturated zone. The approach incorporates models for temporal behaviour of the target into the processing of geophysical data. The method is based on a state space representation of the problem, and estimates are computed using the extended Kalman filter algorithm .

We proposed method to take account for approximation errors in the parameters of evolution model by constructing a statistical model of the differences between the heterogeneous and approximate representations of permeability in fluid flow model. The enhanced evolution model allows for the use of a simplified and computationally efficient evolution model in the nonstationary inversion. Using an evolution model capable of simulating nonisothermal multiphase flow, the proposed approach can also be adopted for the monitoring of geothermal systems

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