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R. McKIBBIN[1]
Professor of Applied Mathematics, Institute of Information and Mathematical Sciences
Massey University, Albany Campus, Auckland

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Full address/phone/fax:

[1]Institute of Information and Mathematical Sciences
Massey University
Private Bag 102 904, NSMC, Auckland

'phone (9) 414 0800 extn 41040
fax (9) 441 8178
e-mail R.McKibbin@massey.ac.nz

MODELLING DEPOSITION OF HYDROTHERMAL ERUPTION EJECTA

R. MCKIBBIN

Massey University, Auckland, New Zealand

SUMMARY – Analytic solutions of the advection-dispersion equation that may be used to describe airborne concentrations after a release of small particles at some height are used to model the deposits from hydrothermal eruption events. The dynamics of the eruption jet are used to estimate the height to which rock particles are lifted, while wind velocity and turbulence parameters allow calculation of the dispersion and deposit profiles of the aerosols and liquid droplets.

The method can also be used for modelling other wind-dispersed particles (e.g. sand, dust, pollen, spray droplets) and can be also be continued below water surfaces to predict seabed or lakebed deposits of land-based solid-particle ejecta.

1. INTRODUCTION

Small particles (sand, dust, droplets, etc.) that are released into the atmosphere, descend under gravity while being dispersed by the wind. Generally the atmosphere does not move uniformly; the wind speed and direction, and also the turbulence length scales, all vary with elevation. By modelling the atmosphere as a flow that is layer-wise uniform, analytical solutions of the conservation equations that describe particle movement through an elevation-varying atmospheric flow may be found. Then, deposition concentrations on the ground may be calculated. While it is possible to numerically calculate solutions to the non-linear equations that result from a continuously-stratified model, the finitely-layered system better reflects the kind of information which is available on air movement. Because the solutions are expressed as explicit formulae, analyses of sensitivity of the deposit structure to the various parameters are easily carried out.

Key assumptions made in the model are as follows:

- the wind is modelled as having a layer-wise uniform velocity (i.e. the wind is constant in direction and speed over each height interval);
- the ground surface is assumed approximately horizontal and the variation of topography does not influence the average transport mechanisms;
- each cohort of material is released at a certain height, where each particle quickly takes up a velocity which corresponds to the wind speed laterally and the particle's "settling speed" downwards (the motion of large rock fragments is not included in this model);
- the turbulence within the air flow is represented by downwind, crosswind and vertical characteristic length scales within each wind layer. Usually, the vertical dispersion is taken to be negligible compared with the vertical advection;
- deposits of different-sized cohorts of particles may be superposed for releases which contain various particle sizes. A hydrothermal eruption contains particles of many sizes; the solids are partitioned into standard sieve diameters, the deposits of which are then combined.

2. THE MODEL

The advection-dispersion equation which describes the motion of a cohort, with total mass Q, of uniformly-sized particles released at the point $(0,0,H)$ a distance H above the Cartesian origin, at time $t = 0$, under the above assumptions, is based on the equation of conservation of mass:

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{q} = Q\delta(x)\delta(y)\delta(z-H)\delta(t) \qquad (1)$$

where the point source term on the RHS is expressed in terms of Dirac delta functions. The volumetric mass concentration $c(x,y,z,t)$ of particles (in kg/m^3) in the atmosphere depends on the specific mass flux $\mathbf{q}$ which is a combination of advection by the wind, turbulent dispersion and settling under gravity. [Initially, the concentration is zero, i.e. $c(x,y,z,0) = 0$.] The mass flux term is taken to be of the form

$$\mathbf{q} = c\mathbf{u} - \underline{\mathbf{D}} \otimes \nabla c - cS\mathbf{k} \qquad (2)$$

where $\mathbf{u} = (U,V,0)$ is the mean wind velocity vector, with wind speed $W = (U^2+V^2)^{1/2}$, while S is the settling speed in the downward direction ($\mathbf{k}$ is a unit vector in the z-direction) and $\underline{\mathbf{D}} = W\underline{\mathbf{L}}$ is the dispersion tensor written in terms of wind speed and a dispersion length tensor. Substitution of (2) into Equation (1) gives an advection-dispersion equation for c:

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\mathbf{u} - \underline{\mathbf{D}} \otimes \nabla c - cS\mathbf{k}) = Q\delta(x)\delta(y)\delta(z-H)\delta(t) \qquad (3)$$

The dispersion tensor is of the general form

$$\underline{\mathbf{D}} = \begin{bmatrix} D_{xx} & D_{xy} & 0 \\ D_{xy} & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \tag{4}$$

If the wind direction is at angle θ to the x-axis (i.e. $U/W = \cos\theta$, $V/W = \sin\theta$), and the longitudinal (downwind) and transverse (crosswind) dispersion coefficients are D_L and D_T respectively, then the second-order symmetric dispersion tensor has the form

$$\underline{\mathbf{D}} = \begin{bmatrix} D_L\cos^2\theta + D_T\sin^2\theta & (D_L - D_T)\sin\theta\cos\theta & 0 \\ (D_L - D_T)\sin\theta\cos\theta & D_L\sin^2\theta + D_T\cos^2\theta & 0 \\ 0 & 0 & D_{zz} \end{bmatrix} \tag{5}$$

Furthermore, if it is assumed that the settling speed, the wind speed and direction, as well as the turbulent dispersion, vary with elevation, then $S = S(z)$, $U = U(z)$, $V = V(z)$ and $\underline{\mathbf{D}} = \underline{\mathbf{D}}(z)$, and Equation (3) may be written as:

$$\begin{aligned} &\frac{\partial c}{\partial t} + U(z)\frac{\partial c}{\partial x} + V(z)\frac{\partial c}{\partial y} - S(z)\frac{\partial c}{\partial z} - \frac{dS}{dz}c \\ &= D_{xx}(z)\frac{\partial^2 c}{\partial x^2} + 2D_{xy}(z)\frac{\partial^2 c}{\partial x \partial y} + D_{yy}(z)\frac{\partial^2 c}{\partial y^2} \\ &+ D_{zz}\frac{\partial^2 c}{\partial z^2} + \frac{dD_{zz}}{dz}\frac{\partial c}{\partial z} + Q\delta(x)\delta(y)\delta(z-H)\delta(t) \end{aligned} \tag{6}$$

In general, this equation cannot be solved analytically, and a full-scale numerical approach is necessary for finding the concentration c and consequent deposits. However, for a uniform airflow in the x-direction with constant speed W and constant downwind (longitudinal) and crosswind (transverse) dispersion coefficients D_L and D_T respectively, and negligible vertical dispersion ($D_{zz} = D_V = 0$), the concentration is given at time t after release by

$$\begin{aligned} c(x,y,z,t) &= \frac{Q}{4\pi\sqrt{D_L D_T t^2}} \cdot \\ &\cdot \exp\left[-\frac{(x-Wt)^2}{4D_L t} - \frac{y^2}{4D_T t}\right]\delta[z-(H-St)] \end{aligned} \tag{7}$$

At time $0 < t < H/S$, the particles are spread laterally at a height $z = H - St$ above the ground. The deposit on the ground is found by integrating the total downward mass flux there:

$$-q_z(x,y,0,t) = \left[D_V\frac{\partial c}{\partial z} + Sc\right]_{z=0} \tag{8}$$

Because the vertical dispersion is effectively zero, the mass deposit is given (in kg/m^2) by

$$\begin{aligned} f(x,y) &= \int_0^\infty Sc(x,y,0,t)dt \\ &= \frac{QS}{4\pi H\sqrt{D_L D_T}}\exp\left[-\frac{(x-UH/S)^2}{4D_L H/S} - \frac{y^2}{4D_T H/S}\right] \end{aligned} \tag{9}$$

In general, the deposit has elliptical level surfaces (contours) centred on the point $(x,y) = (UH/S, 0)$. If $D_T = D_L$, the contours are circular.

3. MULTI–LAYERED MODEL

A piecewise-constant approximation is made to each of the parameters in Equation (6), by dividing the atmosphere into layers. Within Layer i, the wind speed and direction, the particle falling speed and the turbulent dispersion coefficients are taken to be constant, with vertical dispersion negligible. Then, within Layer i, c satisfies the equation:

$$\begin{aligned} &\frac{\partial c}{\partial t} + U_i\frac{\partial c}{\partial x} + V_i\frac{\partial c}{\partial y} - S_i\frac{\partial c}{\partial z} \\ &= D_{Li}\frac{\partial^2 c}{\partial x^2} + D_{Ti}\frac{\partial^2 c}{\partial y^2} + Q\delta(x)\delta(y)\delta(z-H)\delta(t) \end{aligned} \tag{10}$$

with requirements for continuity of concentration and vertical mass flux to be satisfied at the layer interfaces, which are at $z = Z_i$. The particle release height H can be placed in any chosen layer. For the case where the vertical dispersion is small enough to be neglected, analytical solutions can be found.

In the multi-layered model, the particles fall from the release point and the resulting mass distribution that is calculated at the first layer interface [in a form similar to the result given in Equation (9)] is used as a distributed source for the transport through the next layer. In that next layer, the axes are rotated so the "x-axis" is pointing downwind. The process is repeated until the ground is reached, and the resulting deposit is calculated. There is not enough space here to provide the detailed formula for the deposits, but it is easily programmed for direct calculation. Because the mathematical problem is linear, different-sized cohorts can be superposed without interference, and the combined deposit calculated directly. The method can be applied to continuous and time-varying releases, from different positions in the atmosphere.

4. HYDROTHERMAL ERUPTIONS

In general, a hydrothermal eruption consists of a jet of steam, water droplets and small rock particles. While the mechanism that triggers the onset of such events is still a matter of conjecture (seismic disturbance?), they continue to occur irregularly in many geothermal systems around the world. Previous mathematical modelling work by McKibbin, Smith and other workers (2000, 2003, 2005, 2006) has concentrated on both the underground flows and the above-ground mass transport as the eruptions proceed.

In order to make use of the above model, information is needed about the way the particles and droplets in a hydrothermal eruption are ejected into the air; in particular, information about the jet dynamics enables release heights of particular-sized particles to be found. In McKibbin & Smith (2006, in these Proceedings), a formulation for a hydrothermal eruption jet is proposed, and it is that model which is used here.

Height of release from jet

It is assumed that the particles will rise to a height in the jet where their weight is exactly balanced by the upward force (drag) exerted by the hydrothermal eruption column. From McKibbin & Smith (2006), the vertical speed $w(z)$ of the fluid column reduces approximately linearly with elevation from the emerging speed $w(0)$ to zero at the column height $H_{\max}$, i.e.,

$$w(z) \approx w(0)\left(1 - \frac{z}{H_{\max}}\right). \qquad (11)$$

Figure 1 shows a comparison of calculated and approximate jet speeds as functions of height for one particular example, where $w(0) = 20\ \mathrm{ms}^{-1}$.

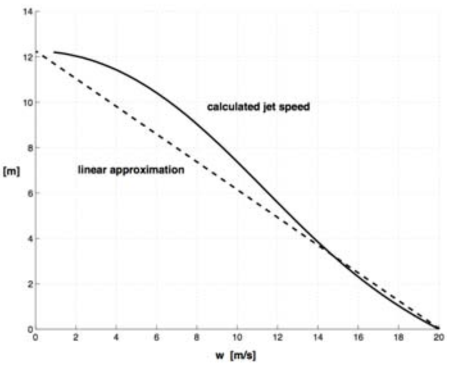

Figure 1: Vertical jet speed w of gas phase at height z: (——) values from Smith & McKibbin (2006); (– – –) linear approximation from Equation (11).

The release height for a particle with settling speed S is therefore given approximately by

$$z = H_{\max}\left(1 - \frac{S}{w(0)}\right) \qquad (12)$$

provided that $w(0) > S$, i.e. that the emerging jet is moving fast enough to lift the particle off the ground.

Settling speed of particles

The settling speed S of a particular particle depends on its size, shape and weight. For particles which are approximately spherical, with diameter d, the requirement that the particle's weight be exactly balanced by the drag exerted by the upward moving jet is expressed by

$$\rho_p \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{1}{2}\rho_a C_D \pi\left(\frac{d}{2}\right)^2 S^2 \qquad (13)$$

where ρ_p and ρ_a are the densities of the particle and air respectively, and the drag coefficient C_D is a function of the Reynolds Number,

$$\mathrm{Re} = \frac{\rho_a d S}{\mu_a}, \qquad (14)$$

where μ_a is the dynamic viscosity of the air. Here, we assume the form (Perry *et al.*, 1984):

$$C_D = \frac{24}{\mathrm{Re}}(1 + 0.14\,\mathrm{Re}^{0.7}) \quad \text{for} \quad \mathrm{Re} < 1000$$

and (15)

$$C_D = 0.447 \quad \text{for} \quad \mathrm{Re} \geq 1000.$$

For a given particle size and density, Equations (13–15) can be solved to give S.

Particle size distribution

In geology and volcanology, grain-size analyses of tephra are made using sieve diameters, which are in turn converted to ϕ-values *via* the formula:

$$d = 2^{-\phi} \quad \text{i.e.} \quad \phi = -\log_2 d \qquad (16)$$

where diameter d is measured in mm (e.g. see Bonadonna & Houghton, 2005). This means that small particles have large ϕ-values, and *vice versa*. Table 1 gives settling speed values for a range of particle sizes measured by standard ϕ-values, for particles with density 1500 kg m^{-3}, and also for water droplets with density 1000 kg m^{-3}. [Note that for $\phi \geq 5$, diameters are given in microns and speeds in mm s^{-1}.]

Table 1: Calculated values of settling speed S for particles of various sizes and densities: rock particles 1500 kg m^{-3}, water drops 1000 kg m^{-3}. [* Water droplets: $d_w \leq 8$ mm, with a maximum speed of about 9 m s^{-1}. See Perry *et al.* (1984).]

ϕ	diameter (□m)	S_p (m s^{-1})	S_w (m s^{-1})
–4	16	24	20*
–3	8	17	14*
–2	4	12	9.8
–1	2	8.5	6.8
0	1	4.9	3.8
1	0.5	2.7	2.0
2	0.25	1.3	0.96
3	0.125	0.51	0.36
4	0.0625	0.16	0.11
	diameter (μm)	S_p (mm s^{-1})	S_w (mm s^{-1})

5	31	43	29
6	16	11	7.4
7	8	2.8	1.9
8	4	0.70	0.46
9	2	0.17	0.11
10	1	0.044	0.029

Use of Equation (12) and Table 1 now allows the release height for each ϕ-class to be calculated. In the example shown in Figure 1, $w(0)$ = 20 m s^{-1} and H_{max} = 12.2 m. Table 2 shows the height in the jet where the weight and drag are equal for each particle size. Note that, in this case, the fine particles with diameter less than 0.25 mm all rise to very near the top of the jet column.

Table 2: Heights in the eruption jet where the nett force is zero, and settling time to ground:
z_p, t_p for rock particles (1500 kg m^{-3});
z_w, t_w for water drops (1000 kg m^{-3}).
[# From Table 1, note that for $\phi = -4$, $S_p > w(0)$.
* See note in caption for Table 1.]]

ϕ	z_p (m)	t_p (s)	z_w (m)	t_w (s)
–4	– #	– #	0.27*	0.014*
–3	1.9	0.11	3.8*	0.27*
–2	4.9	0.41	6.2	0.63
–1	7.0	0.82	8.0	1.2
0	9.2	1.9	9.9	2.6
1	10.6	3.9	11.0	5.5
2	11.4	8.8	11.6	12
3	11.9	23	12.0	33
4	12.1	75	12.1	110
5	12.2	4.7 min	12.2	7.0 min.
6	12.2	18 min.	12.2	27 min.
7	12.2	72 min.	12.2	1.7 min.
8	12.2	4.8 hr	12.2	7.4 hr
9	12.2	20 hr	12.2	31 hr
10	12.2	77 hr	12.2	120 hr

Particle size distribution

The last remaining component required is the actual distribution of the particles amongst the grain-size ϕ-classes. For a given total mass discharge from a hydrothermal eruption jet, particles of all sizes that are small enough to be lifted from the ground will contribute to the eruption. Also, water droplets from the erupting fluid as well as from condensing vapour will be lifted by the jet until they are too large to be elevated any further. It is likely that many will adhere to solid particles and act as a binding agent to aid coalescence of smaller particles into agglomerates which will have a greater settling speed and fall as mud-like particles. Here, to illustrate the method, we will calculate the deposit from an event with a somewhat arbitrary particle size distribution.

The supposed mass distribution amongst different-sized particles is set out in Table 3. If the total volume V of ejected solid material is known, then the total mass M can be calculated from $M = \rho_p V$, where in this case ρ_p = 1500 kg m^{-3}. This is only approximate, as the *in situ* pre-erupted material and that deposited both contain pore space.

The wind

The eruption jet in our example has a maximum height of 12.2 m. We assume that the wind profile is exponential, of the form

$$U = U_f \left(\frac{z}{H_f} \right)^a$$

with a free windspeed of U_f = 5 m s^{-1} at a height of H_f = 20 m, and with a = 0.2.

Table 3: Mass distribution amongst the rock particle size classes for an example eruption with an initial ejection speed at the ground of $w(0)$ = 20 m s^{-1}.
[# Note from Table 1 that for $\phi = -4$, $S_p > w(0)$.]

ϕ	S_p (m s^{-1})	z_p (m)	mass fraction (%)
–4	24	– #	–#
–3	17	1.9	0
–2	**12**	**4.9**	**5**
–1	**8.5**	**7.0**	**10**
0	**4.9**	**9.2**	**10**
1	**2.7**	**10.6**	**15**
2	**1.3**	**11.4**	**15**
3	**0.51**	**11.9**	**20**
4	**0.16**	**12.1**	**15**
5	**0.043**	**12.2**	**10**
6	0.011	12.2	0
7	0.0028	12.2	0
8	0.00070	12.2	0
9	0.00017	12.2	0
10	0.000044	12.2	0
		Total	**100**

We approximate this as a five-layer system with four layers each 5 m thick, topped by a semi-infinite layer; within each layer the wind-speed is uniform (see Figure 2), corresponding to a value mid-way through the layer. The wind direction varies with altitude, but the turbulence length scales are assumed to be uniform; this leads to increasing values of the dispersion coefficients $(D_{Li}, D_{Ti}) = W_i(L_{Li}, L_{Ti})$. The wind parameters are shown in Table 4.

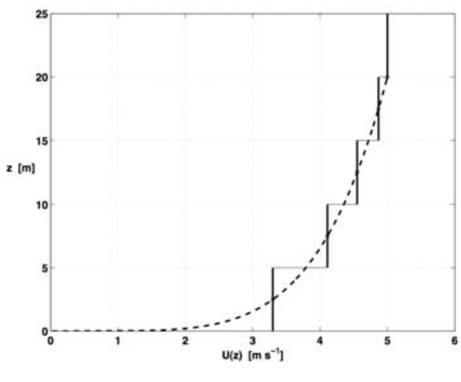

Figure 2. Exponential wind profile and the approximating multi-layered system.

Note that, if required, one can add a layer or layers of water below the wind strata. These layers can be moving to simulate water currents. The settling speeds of particles may be calculated from Equations (13–15) with density and viscosity of water instead of air.

Deposit of solid ejecta

Deposits from each of the eight non-zero masses of grain-size cohorts listed in Table 3 may be superposed. Particles with $\phi = -2$ are released in Layer 5, those with $\phi = -1, 0$ in Layer 4, and the other five sizes $\phi = 1, \ldots, 5$ in Layer 3. Because they are released at greater height and fall more slowly, the spread of the smaller particles is greater than that of the larger ones. Figure 4 shows the pathways of the centre of mass (c.o.m.) of some of the separate grain-size cohorts, as well as the contours of the total mass deposit. The c.o.m. pathways all originate within the eruption jet above its base at $(x, y) = (0, 0)$. Depending on the height to which the particles are lifted, the particles move with the wind, first in the layer in which they leave the jet, and then down through the layers below (if any).

Table 4. Values of interface heights, wind speeds, directions and turbulence length scales for the example case where the wind motion is modelled by a multi-layered system.

Layer	Z_i (m)	W_i (ms^{-1})	θ□	L_{Li} (m)	L_{Ti} (m)
1	20	5.00	0	1	0.5
2	15	4.86	20	1	0.5
3	10	4.55	–20	1	0.5
4	5	4.11	20	1	0.5
5	0	3.30	0	1	0.5

The lighter particles ($\phi = 1, \ldots, 5$) are lifted to Layer 3 ($10 < z < 15$ m) and, because of their slow settling speed (see Table 2), are transported considerable distances before reaching the ground. Because there is then plenty of time for dispersion, these smaller particles are spread more thinly where they land. Figure 5 shows contours of the deposit from the cohort of $\phi = 5$ particles (diameter 31 μm; release height 12.2 m; settling speed 43 mm s^{-1}; time to ground 4.7 minutes).

These particles are transported a long way (of the order of a kilometre or more) downwind of the eruption, and would not be regarded as part of the main ejecta deposit. However, they form part of the fine material that coats the surrounding areas with a thin powder-like coating, some distance from the eruption site.

Further analysis is possible of the distribution of grain-sizes within any sample gathered from the eruption ejecta. At the eruption site, the relative mass fractions of the ϕ-classes are shown in Table 3. The analytic solutions enable the relative mass fractions deposited at any point on the ground to be calculated.

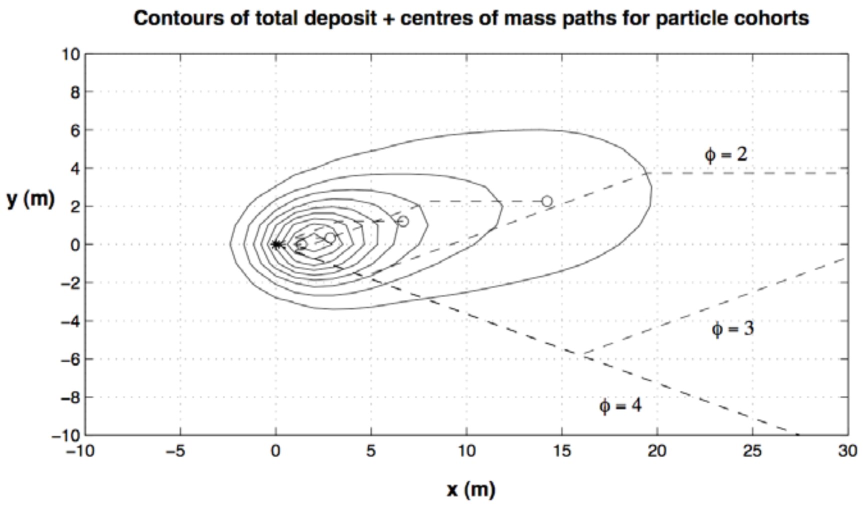

Figure 4. Contours (at 10 % intervals) of the total mass deposit for the example data. The pathways of the centres of mass for the different grain-size cohorts are shown as dashed lines, some of which are marked with their ϕ-values.

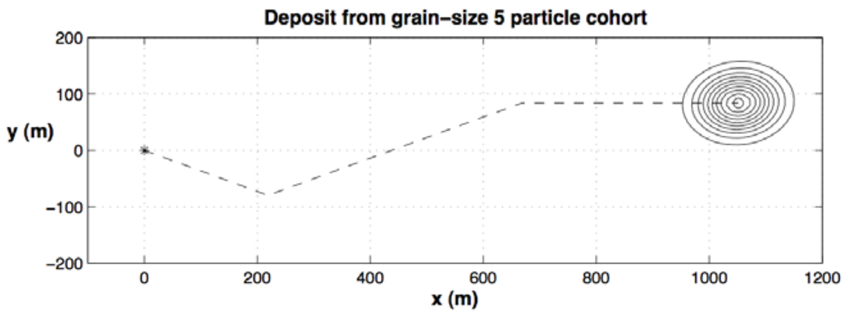

Figure 5. Contours (at 10 % intervals) of the total mass deposit for the $\phi = 5$ cohort.

A profile for a distance of 70 m along $y = 0$ (this is in the direction of the wind at ground level) is shown in Figure 6. The lines show the cumulative mass deposits for the various g□□□□□□□□cohorts with $\phi = -2$ at the bottom up to $\phi = 5$ at the top. The deposits for $\phi > 2$ in this region are too small to distinguish here. The bold curve shows the total deposit $f(x, 0)$, in kg m^{-2}, for $-20 < x < 50$ m.

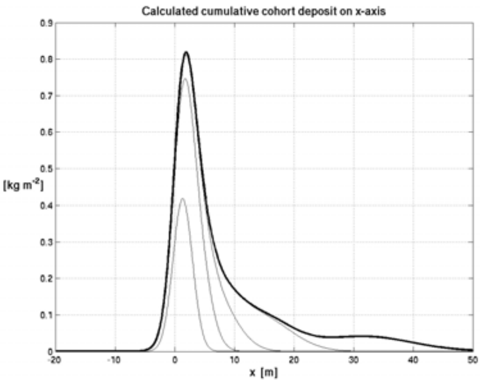

Figure 6. Cumulative mass deposit for the various g□□□□□□□□cohorts with $\phi = -2$ at the bottom up to $\phi = 5$ at the top. The deposits for $\phi > 2$ in this region are too small to distinguish here.

Field-data test?

It is of interest to know what a collected sample from the deposit would show in terms of grain-size composition. This is easily computed by scaling the cumulative values into fractions of the total at each point. For the same data as in Figure 6, these values are shown in Figure 7.

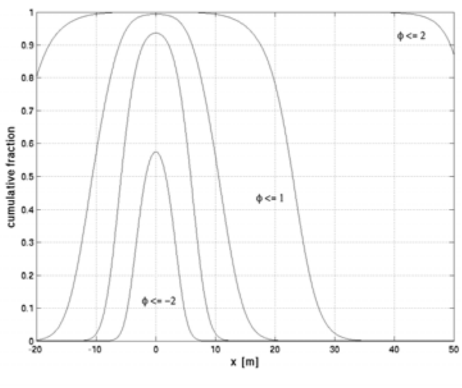

Figure 7. Cumulative mass fractions for the various g□□□□□□□□cohorts with $\phi \leq -2$ at the bottom. The deposits for $\phi > 3$ in this region are too small to distinguish here.

Although the curves appear very symmetrical in this case, it is not so generally. However, Figure 7 shows that, for this particular example, samples collected from, say, 5 m upwind and 5 m down-wind of the eruption vent, should have very nearly the same fractional composition of grain sizes, although, of course, the total mass deposit would be much smaller upwind than downwind (see Figure 6). Is there data available that could be used to test such a calculated result?

SUMMARY

This work is a further development of earlier research into finding analytical expressions for deposits of wind-blown particles released into the atmosphere. The elements required for modelling the transport and deposit of solid material ejected from a hydrothermal eruption have been presented.

Analytical solutions for the advection-dispersion equation have been used to directly calculate how small particles settle to the ground through a non-uniform wind. These have been used to estimate the particulate deposits from an eruption, by using release heights from an eruption jet model.

An example set of mass release data has been used to calculate the particle-size composition of the total deposit. Some field data could be used to test the results. The inverse problem of finding all parameters from field samples would be difficult.

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