MATHEMATICAL MODELLING OF VAPOUR-DOMINATED GEOTHERMAL SYSTEMS

I. PESTOV

PhD student, Mathematics Dept., Victoria University of Wellington, Wellington, NZ

SUMMARY – Dimensional analysis, combined with physical assumptions and the qualitative examination of experimental data, is invoked to bring out further insight into some aspects of mathematical modelling of vapour-dominated geothermal systems such as constraints on vertical heat fluxes and temperature gradients.

1 INTRODUCTION

A number of geothermal systems containing vapour-dominated regions are now known around the world. Examples of such systems include Larderello in Italy, The Geysers in California, Wairakei in New Zealand, Kawah Kamojang in Indonesia, Olkaria in Kenia, Matsukawa in Japan. Conceptual and mathematical models of these systems are well presented in geothermal literature (White *et al.,* 1971; Schubert and Straus, 1981; Pruess *et al.,* 1087; Weir and Young, 1991; Truesdell, 1991; McGuinness *et al.,* 1991; Truesdell *et al.,* 1993).

The process of geothermal modelling is complicated by uncertainty of field data and the enormous number of different effects present in geothermal reservoirs. From this point of view, dimensional analysis can be a useful tool for validation of conceptual models and selection of dominant effects.

In this note, dimensional analysis, combined with physical assumptions and the qualitative examination of experimental data, is invoked. The total number of parameters, describing a vapour-dominated geothermal reservoir, is reduced to several dimensionless quantities. The usefulness of these dimensionless quantities is demonstrated further.

2 DIMENSIONAL ANALYSIS IN MATHEMATICAL MODELLING

The governing conservation equations for two-phase flows in porous media, derived from the basic principles of fluid mechanics and experimental Darcy's law, are given in the work of Cheng (1978). With the help of simplified thermodynamic relationships, suggested for geothermal studies by Grant (1974), these equations can be reduced to two dependent variables in terms of saturation pressure or temperature and one of the saturations. The reduced system of two equations, written in dimensionless form (Pestov, 1993), involves dimensionless parameters, which play a key role in the process of mathematical modelling.

2.1 Key dimensionless parameters

The key dimensionless parameters are:

m–ratio of density of steam to that of water on the hot surface;

$\tilde{m}$–ratio of density of steam to that of water on the cool surface;

R_a–Rayleigh number in porous media saturated by water and steam;

ϵ and δ–latent heat factors;

μ–ratio of dynamic viscosity of steam to that of water.

The corresponding expressions are:

$$m = \frac{\rho_v^*}{\rho_l^*}, \quad \tilde{m} = \frac{\tilde{\rho}_v}{\rho_l^*}, \tag{1a}$$

$$R_a = \frac{\rho_l^* C_{pl} k \rho_l^* g H m}{\alpha \mu_l}, \quad \epsilon = \frac{c}{l^*}, \tag{1b}$$

$$\delta = \frac{C_{pl} T^*}{l^*}, \quad \mu = \frac{\mu_v}{\mu_l}. \tag{1c}$$

Here, ρ_v and ρ_l are densities of vapor and liquid phase respectively, C_{pl} is specific heat of water, k is permeability, g is acceleration due to gravity, H is vertical reservoir dimension, α is effective thermal conductivity, μ_v and μ_l are dynamic viscosities of steam and water respectively (assumed to be constant), c is a coefficient from the state equation of steam $p/\rho_v = c = const$, p is saturation pressure, l is latent heat and T^* is the temperature change across a reservoir (Pestov, 1993). The superscript '$*$' indicates the characteristic quantities and the symbol 'tilde' stands for parameters on the cool surface.

The Rayleigh number R_a enters in the energy equation by dividing the coefficient before the conductive term (Pestov, 1993). We can use the following general definition of the Rayieigh number, given in the book of Sedov (1959):

$$Re_a = \frac{L \Omega C}{\lambda},$$

where L is the characteristic length, Ω is the characteristic velocity, C is the specific heat per unit volume and λ is the coefficient of heat conduction.

For two-phase counterflow the characteristic velocity must include a parameter m. Hence, we choose

$$\Omega = \frac{k\rho_l^* g m}{\mu_l}.$$

This choice of a characteristic velocity defines the Rayleigh number in porous media saturated by water and steam.

The non-dimensional parameters, listed above, can be easily calculated. For a typical vapour-dominated reservoir we take:

$$\rho_l^* = 814\, kg/m^3, \quad g = 9.8\, m/sec^2,$$

$$l^* = 1.76 \times 10^6\, J/kg, \quad H \sim 10^3\, m,$$

$$\mu_l = 10^{-4}\, Nsec/m^2, \quad \mu_v = 2 \times 10^{-5}\, Nsec/m^2,$$

$$k = 10^{-14}\, m^2, \quad c = .2 \times 10^6\, J/kg.$$

$$C_{pl} = 0.46 \times 10^4\, J/kg^\circ C,$$

$$\tilde{T} = 240^\circ C, \quad T^* \sim 10^\circ C,$$

Therefore, we have

$$m \sim 10^{-2}, \quad \tilde{m} \sim 10^{-2},$$

$$\delta \sim 10^{-1}, \quad \epsilon \sim 10^{-1}$$

$$p - 10^{-1}, \quad R_a \sim 10^2.$$

Since the Rayleigh number is large, the coefficient multiplying the conductive term is small. The latter result allows to neglect the conductive component of the total heat flux everywhere except in a thin layer of the order of $\sqrt{\delta/R_a}$ in depth at the top of the flow region.

2.2 Constraints on convective heat flux

Consider one-dimensional steady water-steam counterflow ignoring the thin layer at the top of the flow region, where conductive heat flux is important. Under steady-state conditions, the energy equation can be intergrated and the first integral is:

$$\hat{V}_l \hat{l} = \hat{Q}, \tag{2}$$

where V is a non-dimensional Darcy velocity of the liquid phase and Q is a non-dimensional convective heat flux. ($\hat{Q} = Q/\rho_l^* l^* \Omega$.)

Combining eq. (2) with the non-dimensional form of Darcy's law, obtained by Pestov (1993), gives the following non-dimensional equation for liquid phase relative permeability X (assuming steam relative permeability $Y = 1 - X$) at every point z:

$$X^2 + \left[\frac{\mu - m\hat{\rho}_v}{\hat{\rho}_v \hat{l}(1 - m\hat{\rho}_v)}\hat{Q} - 1\right] X + \frac{m}{\hat{l}(1 - m\hat{\rho}_v)} Q = 0. \tag{3}$$

(The dimensionless quantities above are marked with 'hats'.)

If the roots of eq. (3) are all real and non-negative, then

$$0 \le \hat{Q} \le Q_1, \quad Q_2 \le \hat{Q} \le \infty$$

and

$$Q \le Q_3.$$

Here

$$Q_{1,2} = \frac{\hat{\rho}_v \hat{l}(1 - m\hat{\rho}_v)}{(\mu - m\hat{\rho}_v)^2}\left(\sqrt{\mu} \pm \sqrt{m\hat{\rho}_v}\right),$$

$$Q_3 = \frac{\hat{\rho}_v l(1 - m\hat{\rho}_v)}{(\mu - m\hat{\rho}_v)}.$$

Since Q_2 is greater than Q_3 for any combination of parameters p and m, the only important value is Q_1. Thus, the critical non-dimensional heat flux is:

$$Q_{cr} = \min\{Q_1(z)\} = Q_1(0) = \frac{\tilde{m}(1 - \tilde{m})}{m(\sqrt{\mu} + \sqrt{\tilde{m}})}. \tag{4}$$

The latter result can be useful for determination of boundary conditions in numerical simulations of two-phase flows: the heat flux must not exceed Q_{cr}.

For a typical vapour-dominated reservoir we take

$$m = 0.03, \quad \tilde{m} = 0.02, \quad \mu = 0.2.$$

Eq. (4) gives $Q_{cr} = 1.89$. Returning to dimensional quantities, we have $(Q/k)_{cr} = 6.6\ kw/darcy\ m^2$. In comparison, many geothermal reservoirs have $Q/k = 0.1$ to $0.2\ kw/darcy\ m^2$. The corresponding typical values of Q = 0.03–0.06 are significantly below Q_{cr}.

2.3 Constraints on temperature gradient

This subsection begins with a brief summary of the main results in Pestov (1994), and then expands upon those results.

Consider now the Clausius-Clapeyron equation, which can be written in the form:

$$\frac{dT}{dp} = \epsilon \frac{T(1 - m\hat{\rho}_v/\hat{\rho}_l)}{p\hat{l}} \qquad (5)$$

(Note that here T is the Kelvin temperature.)

If the characteristic density ratio m is small and latent heat varies only slightly, then eq. *(5)* can be integrated and the solution is:

$$\frac{T}{\tilde{T}} = \left(\frac{p}{\tilde{p}}\right)^{\epsilon}. \qquad (6)$$

Comparison of an analytical power fit (6) (solid line) with a numerical approximation by Grant (1974) (broken line) and experimental data from water-steam tables (points) is presented on Fig. 1. For temperatures 130–300°C the power fit (6) matches the experimental temperature-pressure relationship well with a maximum error of 2% at 300°C. It is more accurate than the approximation of Grant (1974) in the temperature range typical of vapour-dominated geothermal systems.

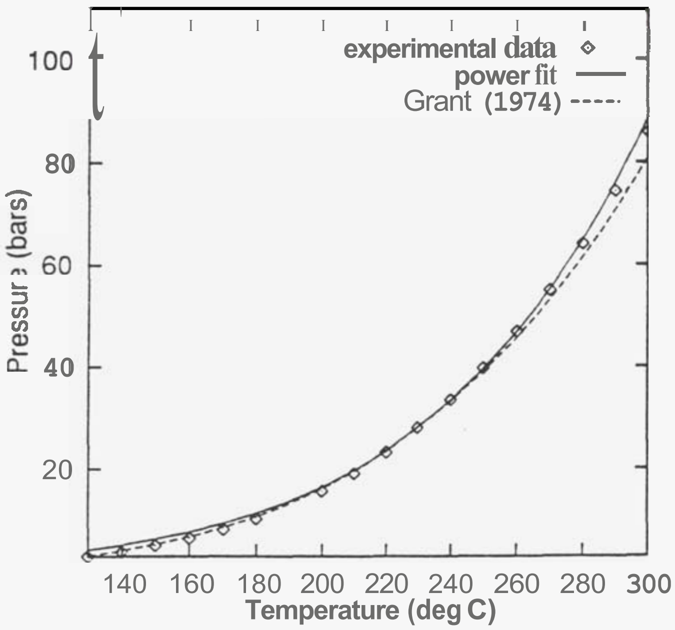

Figure 1. Analytical pressure-temperature curve compared with approximation by Grant **(1974) and experimental data.**

For small ϵ the following expansion of $T/\tilde{T}$ from eq. (6) is valid unless $|\log(p/\tilde{p})|$ is large:

$$\frac{T}{\tilde{T}} = 1 + \epsilon \log \frac{p}{\tilde{p}}. \qquad (7)$$

Since the logarithmic function increases slowly, there is no significant change in temperature until $p/\tilde{p}$ (or the distance z from the top of a region) is large according to eq. (7). The latter signifies that in the upper part of a two-phase region, temperature is a slowly increasing function compared to pressure and temperature gradient is small compared to pressure gradient, provided that parameter ϵ is small.

On Fig. 2. the temperature ratio (broken line), calculated from eq. (7) for a linear pressure fit (solid line), is compared to the temperature ratio, derived from steam tables (points). There is good agreement between analytical and experimental results, supporting the conclusion, that in the upper part of a two-phase zone temperature gradients are **small** compared **to** pressure gradients.

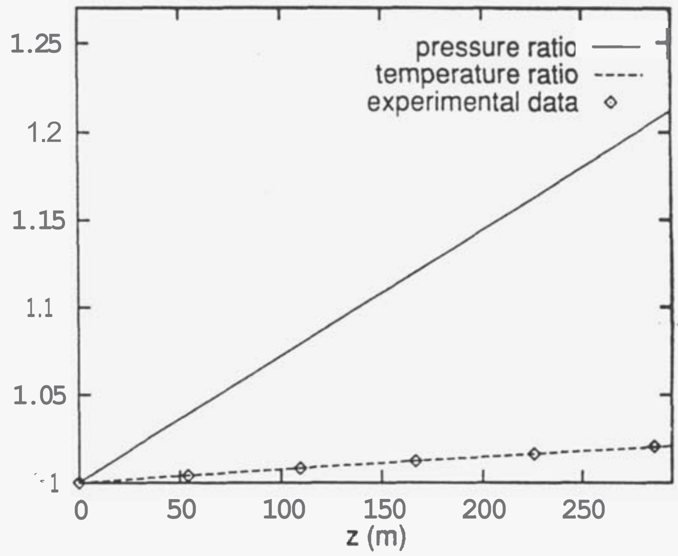

Figure 2. Pressure ratio $p/\tilde{p}$ and temperature ratio $T/\tilde{T}$ versus depth.

3 CONCLUSIONS

An application of dimensional analysis to mathematical modelling of vapour-dominated geothermal reservoirs has been presented. The following conclusions have been drawn:

1. The total number of parameters in the problem considered can be reduced to a substantially smaller number of non-dimensional quantities (1a–1c). Moreover, some of the non-dimensional quantities in (1a–1c) appear to be small or form small coefficients in non-dimensional governing equations. That leads to further simplifications.

2. Since the Rayleigh number is large, the convective component in the total vertical heat flux is dominant and the conductive component is negligible. A vertical heat flux, given in the boundary conditions, must not exceed the

critical value determined by non-dimensional parameters $m, \tilde{m}$ and μ in eq. (4).

3. Since the density ratio m is small and latent heat varies only slightly, the Clausius-Ciapeyron equation can be integrated analytically. For temperatures $130-300°C$, the analytical solution (6) approximates the experimental pressure-temperature relationship well.

4. Given expansion (7) of $T/\tilde{T}$ for small ϵ, temperature gradients are small compared to pressure gradients. This is in agreement with experimental data derived from water-steam tables.

4 ACKNOWLEDGEMENTS

I thank my supervisors Dr. Mark McGuinness and Dr. Graham Weir for their constant attention to my work. Helpful suggestions and comments by Dr. John Burnell, Prof. John Harper, Warwick Kissling, Dr. Steve White and Dr. Roger Young are also appreciated.

5 REFERENCES

Cheng, P. (1978). Heat transfer in geothermal systems. Adv. *Heat Transfer*, Vol. 14, 1-105.

Grant, M.A. (1974). *Simplified Metric Steam Tables for Geothermal work.* DSIR Report *23,* Wellington.

McGuinness, M.J., Blakeley, M., Pruess, K. and O'Sullivan, M.J. (1991). Geothermal Heat Pipe Stability: solution selection by upstreaming and boundary conditions. *Transport in Porous Media,* Vol. 11, 71-100.

Pestov, I. (1993). Mathematical modelling of two-phase geothermal reservoirs. *Proc. 15th New Zealand Geothermal Workshop,* University of Auckland, Auckland, New Zealand, 285-289.

Pestov, I. (1994). On the structure of vapour-dominated geothermal systems. VUW Research Report 94-138, Wellington, New Zealand. Also to appear in: *Proc. 1st Biennial Engineering Mathematics Conference,* Swinburne University of Technology, Melbourne, Australia.

Pruess, K., Celati, R., Calore, C. and Cappetti, G. (1987). On fluid and heat transfer in deep zones of vapor-dominated geothermal reservoirs. Proc. *12th Workshop Geother. Reservoir Engng,* Stanford University, 89-96.

Sedov, L.I. (1959). *Similarity and Dimensional Methods in Mechanics.* Infosearch, London.

Schubert, G. and Straus, J.M. (1981). One-dimensional model of vapor-dominated geothermal systems. *J. Geophys. Res.,* Vol. 86, 9433-9438,

Truesdell, A.H. (1991). The origin of high-temperature zones in vapor-dominated geothermal systems. *Proc. 16th Workshop Geother. Reservoir Engng,* Stanford University, 15-20.

Truesdell, A., Walters, M., Kennedy, M. and Lippmann, M. (1993). An integrated model for the origin of The Geysers. *Geothermal Resources Council Transactions,* Voi. 17, 273-280.

Weir, G.J. and Young, R.M. (1991). Quasi-Steady Vertical Two-Phase Flows in Porous Media. *Water Resources* Research, Vol. 27(6), 1207-1214.

'White, D.E., Muffler, L.J.P. and Truesdell, A.H. (1971). Vapor-dominated hydrothermal systems compared with hot-water systems. *Economic Geology,* Vol. *66,* 75-97.