

Coupled Thermomechanical Boundary Element Modelling (BEM)

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ABSTRACT

As a renewable source of energy, extraction of thermal energy experiments from deep underground hot rock masses and resources is becoming increasingly more feasible, economical and attractive to the research scientists and engineers engaged in this industry. However, because of the harsh environmental conditions, complex coupled hydro-thermo-mechanical behaviour of the rock remains as the main unknown and normally too difficult to determine from direct experiments. That is why analytical and numerical methods such as the BEM and FEM have increasingly been used for such complex coupled boundary value problems (Crouch, 1976; Carter and Alehossein, 1990; Banerjee, 1994; Xiao et al., 1994).

One major issue with most of the BEM research codes is that they can handle either thermal or mechanical and not both in an interactive, coupled manner. Furthermore, one normally needs to develop another interface code to communicate between any two uncoupled codes, otherwise it becomes impossible or too expensive to do it manually. In this paper, a simple practical semi-coupling superposition method of analysis is introduced which can simply be implemented into any interface or even un-coupled BEM code to mimic coupled behaviour of the hot rock. This technique has successfully been implemented into the thermo-coupled boundary element code (BEAN) for simulation of real problems with real geological, geotechnical and geometrical parameters.

BEM for geotechnical and geothermal engineering problems

One major disadvantage of the more familiar numerical methods, i.e. the finite element methods (FEM) and the finite difference and/or distinct element methods (FDM), is that the whole body volume, including the surface boundary of the boundary value problem, needs to be discretised, although solutions for few points might only be required. This expensive volume discretisation problem will be more pronounced when analysing semi-infinite or infinite boundary problems in geotechnical and geothermal engineering. However, Green and Gauss integral theorems inspired mechanical and geotechnical engineers to develop one of the most efficient numerical techniques, i.e. the boundary element method (BEM), where only the bounding surface needs to be discretised. Not only the amount of input data required to describe a problem is greatly reduced this way, but also the influence of the infinite part of the space can automatically be considered in a BEM analysis (Xiao et al., 1994). It is generally more accurate than the similar compatible FEM, by the fact that *discretisation* or *approximations* of the governing equations only occur on the boundary of the problem domain in the BEM. While normally the boundary contains all the problem unknowns, solutions inside the domain, i.e. at the internal points, are normally determined in terms of their corresponding known boundary values. Hence there is no need to solve multiple simultaneous equations for non-boundary or internal-point unknowns. These internal solutions satisfy equilibrium and compatibility equations exactly.

As an application, the excavation ream movement predicted by BEAN for a jointed rock, in Sydney CBD sandstone, was not only impressively close to the field measurement, but also it took orders of magnitude less of the CPU time when compared with those from the similar finite element (FE) and finite difference (FD) models. In particular, the CPU time of the BEM model was only 1/10th of the FE model and 1/500th of the FD model (Alehossein and Carter, 1991).

The numerical solution of most problems in engineering mechanics by the boundary element method is based on the application of the *weighted residual method*, similar to the Galerkin method in the FEM formulation. One major disadvantage of the BEM, when compared to FEM, is that it requires availability of solutions for semi-infinite space or infinite space problems (with no boundaries) due to a unit charge/force/perturbation or the so called the *fundamental* or *Green's function*. As another alternative branch of BEM, the displacement discontinuity BEM (DDBEM) has found an impressive acceptance and attractiveness in geotechnical and geothermal engineering applications (Alehossein and Carter, 1990, 1991; Alehossein, 2000).

Semi-Coupled Thermo-Mechanical BEM

Most practical problems in geomechanics and geothermal engineering are complex and coupled and hence require temporal interactions between various governing equations. A coupled boundary element solution should certainly satisfy all the governing equations valid for each possible individual phenomenon in the coupling process.

Only the application of the method to a simple 1D bar is demonstrated in this paper to prove the method and reveal its salient capability feature. The method can obviously be extended as a tool to analyse any 2D and 3D practical problems in geotechnical and geothermal engineering without any loss of generality. Results of these coupled applications will be published elsewhere.

The governing equations of all the coupling components, together with the BEM formulation, should be satisfied at any time and at any point on the boundary and in the domain of the rock mass. Notice, in the following equations, θ represents temperature, x is coordinate vector, t is time, q is flux or heat flow, u is displacement vector and σ is stress tensor.

Governing Equations:

1. Thermal equilibrium equation
2. Thermo-mechanical equilibrium equation
3. Thermal and flux initial conditions at any given point (i)
4. Thermo-mechanical initial condition
5. Thermal and flux boundary conditions, e.g.

$$\theta_i(x, t) = \bar{\theta}_i(x, t) \quad \text{on the thermal boundary } \Gamma_\theta$$

$$q_i(x, t) = \bar{q}_i(x, t) \quad \text{on the flux boundary } \Gamma_q$$
6. Mechanical displacement and stress boundary conditions

$$u_i(x, t) = \bar{u}_i(x, t) \quad \text{on the displacement boundary } \Gamma_u$$

$$\sigma_i(x, t) = \bar{\sigma}_i(x, t) \quad \text{on the stress boundary } \Gamma_\sigma$$
7. Stress-strain relationships (Hooke's law)
8. Thermal strain (& potential elastic thermal stress) (Hooke's law)
9. Heat conduction-flux-thermal gradient relationships (Fourier Law)
10. Heat conduction-storage-flux-thermal gradient relationships

11. Heat convection-flux-thermal gradient relationships (Newton's law)

Solution procedure for a simple illustrative 1D Example

Two convenient steps are needed to couple thermal with mechanical in any numerical code at any given computational or time step. First, at any given time or time step t , we need to do a pure thermal analysis, using the thermal equations by any numerical code or method (e.g. Finite Difference, Finite Element or Direct or Indirect Boundary Element Method), to determine the thermally induced displacements and stresses everywhere and particularly at the mesh boundaries. In the second step, once we found these thermally induced quantities at the mechanical boundaries, we then solve the thermo-mechanical equations, subject to the same mechanical solutions, but now less the thermal solutions, i.e. we should update our mechanical boundary conditions to the new thermally-induced mechanical boundary conditions.

Consider a simple one dimensional (1D) bar subject to the mechanical boundary displacements and thermal initial and boundary conditions, as depicted in Figure 1.

For simple presentation and quick clarification, a few reasonable assumptions have been made to mainly simplify the mathematical formulations for the sake of proof without any loss of generality.

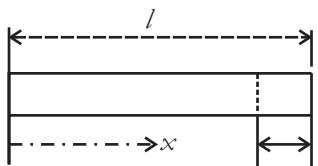
$$\begin{aligned}
 u^*(0, t) &= \bar{u}_0^* = 0 \\
 \theta(0, t) &= \theta_0 e^{-k\alpha^2 t} \\
 \theta(x, 0) &= \cos(\alpha x) - \cot(\alpha l) \sin(\alpha x) \\
 \theta(l, t) &= 0 \\
 u^*(l, t) &= \bar{u}_l^*
 \end{aligned}$$


Figure 1. Coupled thermal and mechanical analysis for a 1D bar.

Applying all the above governing Equations to this simple 1D problem, we have:

$$\frac{\delta^2 u}{\delta x^2} = 4\alpha \frac{\delta \theta}{\delta x} \quad (1)$$

$$\frac{\delta^2 \theta}{\delta x^2} = \frac{1}{k} \frac{\delta \theta}{\delta t} \quad (2)$$

Equation (2) results in the following solution:

$$\theta(x, t) = \theta_0 e^{-k\alpha^2 t} [\cos(\alpha x) - \cot(\alpha l) \sin(\alpha x)] \quad (3)$$

Let's call $\Theta(x, t)$ the x -integral of (3), i.e.

$$\Theta(x, t) = \int_0^x \theta dx = \theta_0 \alpha^{-1} e^{-k\alpha^2 t} [\sin(\alpha x) + \cot(\alpha l) (\cos(\alpha x) - 1)] \quad (4)$$

The value of this function at the points $x=0$ and $x=l$ are as follows:

$$\Theta(0, t) = \Theta_0 = 0 \quad (5)$$

$$\Theta(l, t) = \Theta_l = \theta_0 \alpha^{-1} e^{-k\alpha^2 t} [\csc(\alpha l) - \cot(\alpha l)] \quad (6)$$

Hence, the coupled thermo-elastic displacement solution pertinent to Equation (1) takes the following form:

$$u = u^{\theta} + u^* = \left[4\alpha\Theta(x, t) \right] + \left[\frac{u_l^*}{l} x + u_0^* \right] \quad (7)$$

u^* is the mechanical or the homogeneous solution of Equation (1) and u^{θ} is the particular or the thermally induced component of the displacement.

Therefore, at any given time, t , we can calculate u^{θ} and u^* from Equation (7) and add them together to find the general solution.

However, special treatments are required here if we want to maintain the same *mechanical* boundary conditions at the two points of $x = 0$ and $x = l$. We now verify the solution at these two points:

$$u(0, t) = u_0^{\theta} + u_0^* = 0 + 0 = 0 \quad (8)$$

$$u(l, t) = 4\alpha\Theta_l + u_l^* = u_l^{\theta} + u_l^* \quad (9)$$

Hence, to maintain the same mechanical boundary conditions at the boundary points, we need to solve the homogenous form of Equation (1), but now subjected to the difference between the mechanical and thermal boundary conditions. We now identify this new homogenous solution with two superscripted stars. We can therefore, write:

$$u^{**}(0, t) = u_0^* - u_0^{\theta} \quad (10)$$

$$u^{**}(l, t) = u_l^* - u_l^{\theta} \quad (11)$$

The new homogenous solution is:

$$u^{**}(x, t) = \left[\frac{u_l^* - u_l^{\theta}}{l} x + u_0^* - u_0^{\theta} \right] \quad (12)$$

This approximate solution obtained from the homogeneous solution should be compared with the analytical solution expressed by Equation (7). Indeed, the results from the two methods match impressively well, as shown in Figure 2. For these results the following material properties were assumed in a consistent system of units. $E = 2$, $k_{\text{thermal}} = 1$, $l = 1$, $t = 1$, initial temperature $\theta_0 = 10$, coefficient of thermal expansion $\alpha = 0.785398$, $\Theta_l = 2.235268$, $\Theta_{\text{coeff}} = 16.95334$, $U_{\theta l} = 7.022302$.

In the results of Figure 2, u^{**} corresponds to u^{**} in Equation (12), u^{θ} corresponds to u^{θ} in Equation (7) and u^* corresponds to u^* in Equation (7).

CONCLUSIONS

Application of a simple practical semi-coupling superposition method of analysis was discussed, which can simply be implemented into any interface or even un-coupled BEM code to mimic coupled behaviour of the hot rock in geothermal applications. This technique has successfully been implemented into the thermo-coupled boundary element code (BEAN) for simulation of real problems with real geological, geotechnical and geometrical parameters. In particular, the application of the method to a simple 1D bar is demonstrated in this paper to prove the method and reveal its important modelling feature. The method can be easily extended and generalised for the analysis of more sophisticated and complex 2D and 3D problems in geotechnical and geothermal engineering.

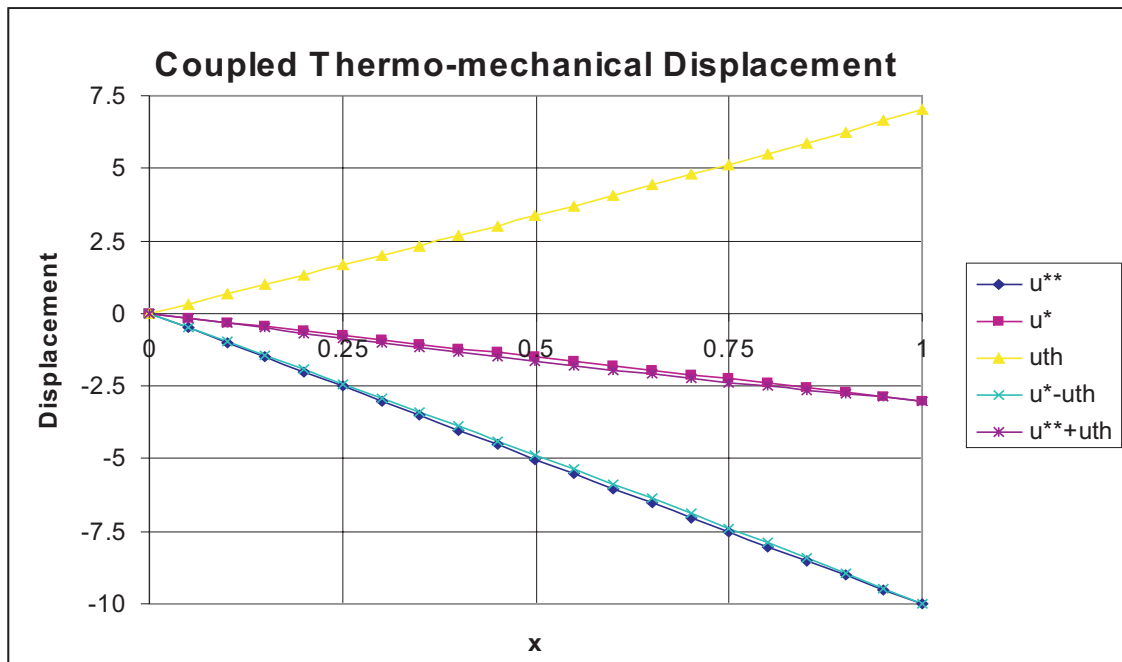


Figure 2. Results of various components of displacement and their combinations for coupled thermo-elastic analysis.

REFERENCES

- Alehossein, H.A. 2000. Computer program **BEAN** Boundary Element Analysis of Anisotropic Media.
- Alehossein, H. and Carter, J.P. 1991. Boundary Element Analysis of Excavations in Aeolotropic Materials. *Proc. 7th Int. Conf. on Computer Methods and Advances in Geomech. (IACMAG-91)*, Cairns, 293-298.
- Alehossein, H. and Carter, J.P. 1990. Elastic Solutions for Two Railway Tunnels Near Excavations in Anisotropic Materials. *Proc. 12th Int. Conf. on Boundary Element Methods*, Sapporo, Hokkaido, Japan.
- Banerjee, P. K. 1994. *The Boundary Element Methods in Engineering*. McGraw Hill New York.
- Carter, J.P., Alehossein, H., Booker, J.R. and Balaam, N.P. 1990. Elastic Solutions for Tunnels near Excavations. *J. Civil Engineering Transactions, Institution of Engineers, Australia*, CE **32**, No.2, 75-86.
- Carter, J.P. and Alehossein, H. 1990. Analysis of Tunnel Distortion Due to An Open Excavation In Jointed Rock. *Computers and Geotechnics*, **9**, 209-231.
- Crouch, S. L. 1976. Solution of plane elasticity problems by the displacement discontinuity method, parts 1 and 2. *Int J for Num Meth in Eng.* **10**, 301-343.
- Xiao, B., Carter, J.P. and Alehossein, H. 1994. "Elastoplastic Analysis of Jointed Rocks Using a Coupled Finite Element and Boundary Element Method". *Int. Jnl. for Numerical & Analytical Methods in Geomechanics*, **4/18**, 445-466.